# Applications of the semisimple tensor product of MV-algebras

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Since the real interval $[0,1]$ is closed to the product operation, a natural problem was to find a complete axiomatization for the variety generated by the standard MV-algebra $([0,1],\oplus,\neg,0)$ endowed with the real product. If the product operation is defined as a bilinear function $\cdot:[0,1]\times[0,1]\to[0,1]$ then the standard model is $([0,1],\oplus,\cdot,\neg,0)$; if the product is a scalar multiplication then the standard model is $([0,1],\oplus,\{r\mid r\in[0,1]\},\neg,0)$, where the function $x\mapsto rx$ is a linear for any $r\in[0,1]$. The approach based on the internal binary product led to the notion of *PMV-algebra* [1, 6, 7], while the approach based on the scalar multiplication led to the notion of *Riesz MV-algebra* [2]. In [4] we defined the *fMV-algebras* adding both an internal product and a scalar multiplication; in this case, our standard model is $([0,1],\oplus,\cdot,\{r\mid r\in[0,1]\},\neg,0)$. The varieties of MV-algebras and Riesz MV-algebras are generated by their corresponding standard models; this is not the case for PMV-algebras and $f$MV-algebras: in these classes the standard models generate proper sub-varieties.

There are obvious forgetful functors between the above mentioned classes of structures:

$$\begin{tikzcd}
\mathbf{MV} & \mathbf{PMV} \arrow[l, "U"'] \\
\mathbf{RMV} \arrow[u, "U"] & \mathbf{fMV} \arrow[l, "U"] \arrow[u, "U"'] \arrow[ul, "U"']
\end{tikzcd}$$

Our goal is to define the left adjoint functors. We do this in [3] for the corresponding subclasses of semisimple structures using the semisimple MV-algebraic tensor product defined in [8] and its scalar extension property [5]. Note that all algebras with product are considered unital. Of particular importance is the *semisimple tensor PMV-algebra of an MV-algebra*, whose construction is inspired by the well-known definition of the tensor algebra. The main result asserts that the following diagram is commutative:

$$\begin{tikzcd}
\mathbf{MV_{ss}} \arrow[r, bend left, "\mathbf{T}"] \arrow[d, bend right, "\mathcal{T}_{\otimes}"'] & \mathbf{PMV_{ss}} \arrow[l, "U"] \arrow[d, bend left, "\mathcal{F}_{\otimes}"] \\
\mathbf{RMV_{ss}} \arrow[u, "U"'] \arrow[r, bend right, "\mathcal{F}_{\mathbf{T}}"'] & \mathbf{fMV_{ss}} \arrow[l, "U"'] \arrow[u, "U"] \arrow[ul, "U"']
\end{tikzcd}$$

As consequence we prove the amalgamation property for semisimple PMV-algebras, semisimple Riesz MV-algebras and semisimple $f$MV-algebras.

# References

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