From Freudental's Spectral Theorem to projectable hulls of unital Archimedean lattice-groups, through compactification of minimal spectra Part 1: the compact case.

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This is the first part of a series of two abstracts, the second one being by Daniel McNeill.

In its basic version, Freudenthal's Spectral Theorem [4] asserts that any element of a Riesz space R with a strong unit u and the principal projection property may be uniformly approximated, in the norm that u induces on R, by abstract characteristic functions – "components of the unit u". Freudenthal's theorem led to a considerable amount of research on Riesz spaces and their generalisations, the lattice-ordered Abelian groups that concern us here. (For background on Riesz spaces and ℓ-groups see [6] and [3, 5].) One main line of research concentrated on extending one given structure G to a minimal completion that enjoys the principal projection property, where Freudenthal's theorem therefore applies. Such an extension is called *the projectable hull* of G. Here we present a new construction of the projectable hull of an Archimedean ℓ-group equipped with a strong order unit u that does not use direct limits, nor essential closures. Our construction exposes instead the intimate connection between projectable hulls and zero-dimensional compactifications of spectral spaces of minimal prime ideals.

In this first talk, we give the basic definitions concerning the notions involved in the construction. We topologize both the space of the maximal prime ideals $\operatorname{Max} G$ and the space of the minimal prime ideals $\operatorname{Min} G$ using the *spectral topology.* The closed sets for this topology are given by subsets of the form

$$\mathbb{V}_M(A) := \{\mathfrak{m} \in \operatorname{Max} G \colon A \subseteq \mathfrak{m}\}$$

and

$$\mathbb{V}_m(A) := \{\mathfrak{p} \in \operatorname{Min} G \colon A \subseteq \mathfrak{p}\},$$

as A ranges over arbitrary subsets of G. The space $\operatorname{Max} G$ is a compact Hausdorff space; the space $\operatorname{Min} G$ is a Hausdorff zero-dimensional space that need not be compact, [1].

It is well known that $\operatorname{Min} G$ is canonically thrown onto $\operatorname{Max} G$, as follows. Given $\mathfrak{p} \in \operatorname{Min} G$, a standard argument shows that, by virtue of the presence of the (strong) unit u, there exists at least one $\mathfrak{m}_\mathfrak{p} \in \operatorname{Max} G$ such that $\mathfrak{p} \subseteq \mathfrak{m}_\mathfrak{p}$. Since the prime ideals of G form a root system under set-theoretic inclusion, such an $\mathfrak{m}_\mathfrak{p}$ must be unique. Hence there is a continuous surjection $\lambda\colon \operatorname{Min} G \twoheadrightarrow \operatorname{Max} G$ defined by

$$\lambda\colon \mathfrak{p} \mapsto \mathfrak{m}_\mathfrak{p}.$$

The composition of λ with the Yosida representation of G ($\widehat{\cdot}\colon G \hookrightarrow \mathrm{C}(\mathrm{Max}\, G)$, [8]) embeds G as a unital ℓ-subgroup into the ℓ-group $\mathrm{C}(\mathrm{Min}\, G)$ of continuous functions $\mathrm{Min}\, G \to \mathbb{R}$ under pointwise operations: $g \in G$ is sent to the function $\widehat{g} \circ \lambda \colon \mathfrak{p} \in \mathrm{Min}\, G \mapsto (\widehat{g} \circ \lambda)(\mathfrak{p}) = \widehat{g}(\mathfrak{m}_{\mathfrak{p}}) \in \mathbb{R}$.

We focus now our attention on the case where $\mathrm{Min}\, G$ is compact. (Equivalent conditions to compactness are given in [2, 7].) Here, the base $\{\mathbb{V}_m(g)\}_{g\in G}$ for the closed sets of $\mathrm{Min}\, G$ is a Boolean algebra under set-theoretic union, intersection and complementation. This base also coincides with the set of all clopen subsets of $\mathrm{Min}\, G$. Let $\mathrm{K}(\mathrm{Min}\, G)$ be the collection of all characteristic functions on $\mathrm{Min}\, G$ – i.e. continuous maps $\mathrm{Min}(G) \to \mathbb{R}$ whose range is contained in $\{0, 1\}$. Then the elements in $\mathrm{K}(\mathrm{Min}\, G)$ are precisely those functions whose value is 1 at each point of some $\mathbb{V}_m(g)$ (with $g \in G$), and 0 otherwise. Let $\widetilde{G}$ be the image of G under the map $\widehat{\cdot} \circ \lambda$, and let $\mathcal{P}(G)$ be the ℓ-subgroup of $\mathrm{C}(\mathrm{Min}\, G)$ generated by $\widetilde{G} \cup \mathrm{K}(\mathrm{Min}\, G)$. Our main result is the following:

Theorem. *The embedding*

$$\begin{aligned} \pi\colon G &\hookrightarrow \mathcal{P}(G) \\ g &\mapsto \widehat{g} \circ \lambda \end{aligned}$$

is the projectable hull of G.

For the generalization of our result to the case where $\mathrm{Min}\, G$ is not compact, please see Daniel McNeill's abstract.

References

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