# A(nother) duality for the whole variety of MV-algebras

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Given a category $C$ one can form its *ind-completion* by taking all formal directed colimits of objects in $C$. The "correct" arrows to consider are then families of some special equivalence classes of arrows in $C$ ([1, V.1.2, pag. 225]). The *pro-completion* is formed dually by taking all formal directed limits. For general reasons, the ind-completion of a category $C$ is dually equivalent to the *pro*-completion of the dual category $C^{\mathrm{op}}$.

$$\text{ind-}C \simeq (\text{pro-}(C^{\mathrm{op}}))^{\mathrm{op}}. \tag{1}$$

Ind- and pro- completions are very useful objects (as they are closed under directed (co)limits) but cumbersome to use, because of the involved definitions of arrows between objects. We prove that if $C$ is an algebraic category, then the situation considerably simplifies.

If $V$ is any variety of algebras, one can think of any algebra $A$ in $V$ as colimit of finitely presented algebras as follows. Consider a presentation of $A$ i.e., a cardinal $\mu$ and a congruence $\theta$ on the free $\mu$-generated algebra $\mathcal{F}(\mu)$ such that $A \cong \mathcal{F}(\mu)/\theta$. Now, consider the set $F(\theta)$ of all finitely generated congruences contained in $\theta$, this gives a directed diagram in which the objects are the finitely presented algebras of the form $\mathcal{F}(n)/\theta_i$ where $\theta_i \in F(\theta)$ and $X_1, ..., X_n$ are the free generators occurring in $\theta_i$. It is straightforward to see that this diagram is directed, for if $\mathcal{F}(m)/\theta_1$ and $\mathcal{F}(n)/\theta_2$ are in the diagram, then both map into $\mathcal{F}(m+n)/\langle\theta_1 \uplus \theta_2\rangle$, where $\langle\theta_1 \uplus \theta_2\rangle$ is the congruence generated by the disjoint union of $\theta_1$ and $\theta_2$. Now, the colimit of such a diagram is exactly $A$. Denoting by $V_{\mathrm{fp}}$ the full subcategory of $V$ of finitely presented objects, the above reasoning entails

$$V \simeq \text{ind-}V_{\mathrm{fp}}. \tag{2}$$

We apply the above machinery to the special case where $V$ is the class of MV-algebras. One can then combine the duality between finitely presented MV-algebras and the category $P_{\mathbb{Z}}$ of rational polyhedra with $\mathbb{Z}$-maps [2], with (1) and (2) to obtain,

$$MV \simeq \text{ind-}MV_{\mathrm{fp}} \simeq \text{pro-}(P_{\mathbb{Z}})^{\mathrm{op}}. \tag{3}$$

This gives a categorical duality for the whole class of MV-algebras whose geometric content may be more transparent than other dualities in literature. In increasing order of complexity one has that an MV-algebra $A$:

1. is dual to a polyhedron (if $A$ is finitely presented);
2. is dual to an intersection of polyhedra (if $A$ is semisimple);
3. is dual to a countable nested sequence of polyhedra (if $A$ is finitely generated);
4. is dual to the directed limit of a family of polyhedra. (if $A$ is in none of the above cases).

## References

[1] P. T. Johnstone. *Stone spaces.* Cambridge Univ. Pr., 1986.

[2] V. Marra and L. Spada. The dual adjunction between MV-algebras and Tychonoff spaces. *Studia Logica (Special issue dedicated to the memory of Leo Esakia)*, 100(1-2):253–278, 2012.