Universita' degli Studi di Firenze
Dipartimento di Statistica "Giuseppe Parenti"

Dottorato in Statistica Applicata
XVIII ciclo

ESTIMATING THE DIFFUSION PART OF THE COVARIATION BETWEEN TWO STOCHASTIC VOLATILITY MODELS WITH LÉVY JUMPS

Fabio Gobbi

Supervisor
Prof. Cecilia Mancini

Advisor
Prof. Giorgio Calzolari

Director of Graduate Studies
Prof. Fabrizia Mealli

Acknowledgements

”Cos’e’ la fortuna? E’ credere di essere fortunati. Ecco cos’e’” Questa frase pronunciata con un’indimenticabile intonazione da Marlon Brando nel film ”A streetcar named desire” mi pare possa rappresentare il mio stato d’animo in questi ultimi 2 anni. Quando supponi di essere fortunato non perdi mai l’ottimismo, i periodi di difficolta’ sono temporanei per definizione, le risorse per continuare non si esauriscono. Si instaura il circolo vizioso per eccellenza, e ti persuadi a vivere nell’unica maniera ammissibile: giorno per giorno, prendendo sul serio solo cio’ che serio. Solo con questa convinzione riesco a spiegarmi per quale ragione il lavoro che si trova in queste pagine ha trovato una conlclusione dignitosa. Io non ho fatto niente di veramente buono nella mia esistenza se non quello di sopravvivere con onesta’ intellettuale (o quella che credo sia l’onesta’ intellettuale), percio’ ritengo di non meritare la fortuna che ho avuta di incontrare le persone che ho incontrato in questi mesi e che hanno contribuito in varia misura alla redazione di questa ricerca e soprattutto alla crescita della mia conoscenza. Un concentrato di buona gente, direi. Prima fra tutte la Prof. Cecilia Mancini, che un giorno di Marzo e’ stata cosi’ gentile da ricevermi nel suo ufficio per discutere di una questione relativa alla convergenza di certi integrali. Non immaginava che da quel momento l’avrei importunata almeno una volta a settimana.....Grazie sinceramente per avermi sopportato....questo lavoro e’ in grande parte merito suo.... Grazie al Prof. Giorgio Calzolari, per la bellissima esperienza vissuta qualche tempo fa a Bertinoro e per possedere un’ironia cosi’ semplice da sembrare antica e un sorriso tra i piu’ contagiosi che conosca.....Grazie al Prof. Pietro Rigo per avermi insegnato nelle sue lezioni e nei nostri incontri la formalizzazione rigorosa alla base del ragionamento astratto e per la sua straordinaria capacita’ di rendere intuitivi concetti decisamente complessi....Grazie al Prof. Gabriele Fiorentini per il suo disincanto e per la sincerita’ con cui si rivolge ad ogni ambiente.....Grazie ai miei compagni di dottorato, naturalmente la parte piu’ consistente di questi tre anni, con i quali ho condiviso la passione per la ricerca e molti momenti di ilarita’ esagerata che anche sforzandomi non potrei dimenticare. A loro va il mio augurio di trovare una strada che sia quella che desiderano. Due di loro meritano una pensiero a parte. Monia, dagli occhi splendenti, per la sua dolcezza (che non penso di meritare), Andrea per avere lasciato troppo presto questa vita.....Infine, un grazie al Prof. Renato Leoni, il quale da diversi anni, per ragioni che ancora mi sforzo di ricercare, mi ha trattato come un amico concedendomi piacevoli conversazioni di scienza e politica e trasmettendomi una sensazione di grande dignita’.

"Hai scoperto di non essere un uomo d'affari dunque....."
"Solo un uomo....."
"Una razza vecchia....."

Contents

1 Introduction and Motivations **1**
- 1.1 General Motivations: financial modelling with jumps . . . 1
- 1.2 Dependence and Lévy processes . . . 2
- 1.3 Financial market, pricing of claims and model uncertainty . . . 3
- 1.4 The Realized Volatility and the Threshold Estimator . . . 5
- 1.5 Outline . . . 6

2 Elements of general theory of stochastic processes **7**
- 2.1 Stochastic processes and Martingales . . . 7
- 2.2 Stochastic Integrals . . . 11
  - 2.2.1 Stochastic integral with respect to a continuous square integrable martingale 11
  - 2.2.2 Stochastic integral with respect to a semimartingale . . . 16
- 2.3 Quadratic Variation and Quadratic Covariation . . . 19
- 2.4 Poisson processes, Lévy processes and Lévy measures . . . 21

3 Preliminary results in continuous case **29**
- 3.1 Introduction . . . 29
- 3.2 Some preliminary lemmas . . . 29
- 3.3 Main results in case of diffusion processes . . . 34
- 3.4 The presence of the drift term . . . 39

4 Main results **43**
- 4.1 Introduction . . . 43
- 4.2 Finite activity case . . . 44
  - 4.2.1 Consistency . . . 44
  - 4.2.2 Central Limit Theorem . . . 47
- 4.3 Infinite activity case . . . 48
  - 4.3.1 Consistency . . . 49
  - 4.3.2 Work in progress: the Central Limit Theorem . . . 68

5 Simulation results **75**
- 5.1 First case: the jump component is a Compound Poisson process . . . 75
- 5.2 Second case: the jump component is a Variance Gamma process . . . 76
- 5.3 Final remarks . . . 77

Chapter 1

Introduction and Motivations

"Try not to become a man of success but rather
to become a man of value"
(A. Einstein)

1.1 General Motivations: financial modelling with jumps

The Wiener process (or the Brownian motion) is the most widely used stochastic process in Mathematical Finance, when we study the time evolution of price, or log-price, of a financial asset. Its fundamental properties are

- Independent and stationary increments;
- Gaussian distribution of increments;
- Continuity (as a function of time) of the sample paths.

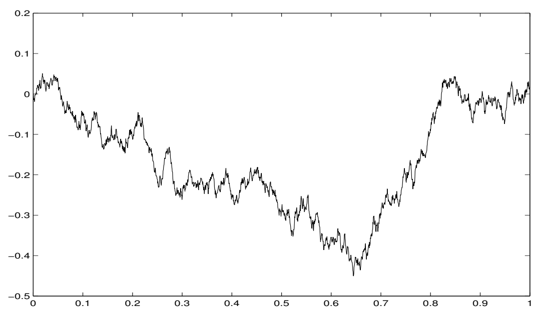

Figure 1.1: Example of a sample path of a Wiener process

However, the analysis of the evolution of a stock price for any asset shows the presence of several discontinuities (jumps) which characterize the price trajectories during a fixed period of time. More precisely, we compare asset returns, that is, the time variation of prices, which are widely dispersed in their amplitude and large peaks (generated by jumps in prices) are frequent, while the increments of the Wiener process, being of Normal type, always have the same amplitude. Moreover, another property of the Wiener paths, the scale invariance, seems not to be verified by the evolution of real stock prices. The scale invariance is the property for which the statistical properties of the Brownian motion are the same at all time resolutions. This is not certainly the case when we consider empirical financial time series. As Cont and Tankov observe *prices moves essentially by jumps* (Cont-Tankov, 2004, p.3). Jump is the key word. It represents the most important motivation that we lead to consider models with discontinuous sample paths. A diffusion model cannot generate sudden discontinuous moves in prices: tail events are the accumulation of many small moves (Cont-Tankov, 2004, p.7). These dissertations drive us in the choice of the most appropriate stochastic volatility model for our purposes, which has the following general form

$$dX_t = a_t dt + \sigma_t dW_t + dJ_t,$$

where $X = (X_t)_{t\in[0,T]}$ is the stochastic process which rule the evolution of the asset price, $a_t dt + \sigma_t dW_t$ is the diffusion part and dJ_t is a jump process which introduces the sudden discontinuities. Two type of jumps can be considered: rare and large jumps and infinite small jumps in every finite time interval. In the first case we speak of *finite activity models*. Here, the jump part is typically a *Compound Poisson process* with finitely many jumps in each finite time interval introduced for the first time by Merton (Merton, 1976). In the second case we speak of *infinite activity models*. Here the jumps itself give the evolution of the process, i.e., the dynamics of jumps is rich enough to generate nontrivial small time behavior (Carr, Geman, Madan, Yor, 2002).

There are other (even if less important) motivations for departing from Gaussian models in Finance which derive by the observation the empirical characteristics of asset returns; in particular, we think of the statistical properties of returns such as heavy tails and absence of (linear) autocorrelations. Moreover, these properties seem to change with the time scale. For example, when microstructure effects come into play (for small intraday time scale), autocorrelations can be significantly different from zero and heavy-tailed feature is less pronounced as the time horizon is increased. See Cont (2001) and Pagan (1996) for more details. When we adopt a Lévy process as a model to drive the price dynamics, the problem of heavy tails can be overcome because the distributions of any Lévy process have positive kurtosis: therefore, the probability of occurrence of large market movements won't be negligible, unlikely the Gaussian case. Besides, the absence of autocorrelation in increments is a property of all Lévy processes.

1.2 Dependence and Lévy processes

In this work, we are interested in studying the dependence between two semimartingales representing the underling stochastic processes which describe the dynamics of stock prices. In particular, we will assume a pair of Lévy processes $(X_t^{(1)})_{t\in[0,T]}$ and $(X_t^{(2)})_{t\in[0,T]}$, which can be decomposed into the sum of three independent parts: the first one of continuous type (C), the second one of finite activity jump type (FA), the third one of infinite activity jump type (IA), i.e.,

$$X_t^{(1)} = C_t^{(1)} + FA_t^{(1)} + IA_t^{(1)},$$
$$X_t^{(2)} = C_t^{(2)} + FA_t^{(2)} + IA_t^{(2)}.$$

Clearly, the joint evolution of $(X^{(1)})_t)_{t\in[0,T]}$ and $(X_t^{(2)})_{t\in[0,T]}$ depends on how several parts are correlated. In the case of random processes the quantity which establishes the covariation is the *quadratic covariation process* (chapter 2, section 3) that is estimated in different ways depending on the assumptions relative to the stochastic model. When we model market movements by Lévy processes, the quadratic covariation is the sum of the dependence of the continuous parts and the dependence of jumps. In the last case the crucial role is played by the simultaneous jumps whose study represents an important future development. Indeed, in this work we concentrate on the covariation between the continuous components $C^{(1)}$ and $C^{(2)}$ and in particular we estimate the quadratic covariation between them, $[C^{(1)}, C^{(2)}]_T$. The most important case is when $C^{(1)}$ and $C^{(2)}$ are of Gaussian diffusion type, that is,

$$dC_t^{(q)} = a_t^{(q)} dt + \sigma^{(q)} dW_t^{(q)}, \quad q = 1, 2,$$

where $a^{(q)}$ are the mean processes, $\sigma^{(q)}$ are the volatility processes and $W^{(q)}$ are two correlated Wiener processes. Here, $[C^{(1)}, C^{(2)}]_T = \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}$ where ρ is the correlation coefficient between $W^{(1)}$ and $W^{(2)}$ which, without loss of generality, is assumed independent of the variable t.

1.3 Financial market, pricing of claims and model uncertainty

We give a brief description of the concepts of Mathematical Finance which serve to understand the importance of the estimate of the integrated volatility and integrated covariance. We essentially follow Klebaner (1999) and Bjork (1998). Let us consider a financial market consisting of several type of financial derivatives and assets such as stocks and bonds. A financial derivative is a contract that allows purchase or sale of an asset in the future on terms that are specified in the contract. *Call option* and *put option* are the most important examples of contingent claims. A call option on stock is a contract gives its holder the right to buy this stock in the future at (predefined) price K. Suppose the period of time is finite $[0,T]$: T represent the time at which the holder can exercise the option. Denote by $(S_t)_{t\in[0,T]}$ the **price dynamics** of the asset. Since $(S_t)_{t\in[0,T]}$ is a stochastic process for mathematical reasons we suppose that a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},P)$ is given. Clearly, if $S_T < K$ the holder will not exercise the option, as he can buy stock cheaper than K, thus this option has value 0. On the contrary, if $S_T \geq K$ then the holder can buy the stock (more realistically shares of stocks) at price K and sell it for S_T making a profit of $S_T - K$. Call options that can be exercised at a fixed time date in the future are known as *European call options* and its value at time T will be

$$C_T = \max(0, S_T - K) = (S_T - K)^+ = (S_T - K)1_{\{(S_T-K)\geq 0\}}.$$

If the value of the claim is specified in the contract, it is not so for the stock, whose value at time T is uncertain and determined by the stochastic evolution of its price. So, the problem of choosing an efficient price for the claim it is crucial to manage financial risk: the theory of pricing of claims (which will be defined below) deal with type of problems.

Definition 1.3.1. *Consider a financial market where time is divided into periods of length h and where trading only takes place at the discrete points T/n, $n = 1,2,....$. Suppose we have an m-dimensional price process $(\boldsymbol{S}_t)_{t\in[0,T]} = (S_{1t},....,S_{mt})_{t\in[0,T]}$, where S_{jt} $j = 1,2,...,m$ are the price processes of different stocks. Define a* ***portfolio strategy*** *(or simply a portfolio) any $\mathcal{F}_t$-adapted m-dimensional process $(\Pi_t)_{t\in[0,T]} = (q_{1t},...,q_{mt})_{t\in[0,T]}$. The value of the portfolio at time t will be given by $V_t = \sum_{j=1}^m q_{jt}S_{jt}$. The process $(V_t)_{t\in[0,T]}$ is called the* value process.

A portfolio is *self-financing* if all the changes in the portfolio are due to gain realized on investment, i.e., no funds are borrowed or withdrawn from the portfolio at any time t (Klebaner, 1999). Moreover, it is called *admissible* if it is self-financing and the corresponding value process is nonnegative. A contingent claim X is a nonnegative r.v. defined on $(\Omega,\mathcal{F}_T)$, and represents an agreement which pays X at time T. For example, for a call option with price K, $X = (S_T - K)^+$. It is called attainable if there exists an admissible portfolio such that $V_t \neq 0$ and $V_T = X$. Now, let α_t be the price at time t of an investment without risk. Then we can define the discounted price process by $Z_t = \frac{S_t}{\alpha_t}$. If there exists a probability measure Q such that the process Z_t is a Q-martingale, such a probability is called the *martingale equivalent measure*. A known result (see for example Klebaner, 1999, p.258) says that if the market is arbitrage free (that is, does not exist a strategy that allows to make profit out of nothing without taking any risk) then there exists a probability measure Q equivalent to P such that the discounted stock process Z_t is a martingale under Q. The most important result about pricing of claims is the following.

Theorem 1.3.2. *In case of arbitrage free market the price C_t of a claim X at time $t \leq T$ is given by V_t, the value of any admissible portfolio replicating X, and*

$$C_t = E^Q(\frac{X}{\alpha_t}|\mathcal{F}_t).$$

In the case of call options we then have $C_t = E^Q(\frac{(S_T-K)^+}{\alpha_t}|\mathcal{F}_t)$. This result suggests to us how to price any attainable claim. When all the claims are attainable, market models are called **complete**. A well known result, which can be found in Harrison and Kreps (1979), says that *the market model is complete if and only if the martingale probability measure is unique.*

The time evolution of stocks price can be described by a stochastic differential equation of diffusion type. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$ be a filtered probability space. Thus, in a diffusion model the price dynamics is assumed to satisfy

$$dS_t = a(S_t)dt + \sigma(S_t)dW_t,$$

where $W = (W_t)_{t\in[0,T]}$ is a Wiener process defined with respect to P. Moreover the process α_t is deterministic and continuous

$$\alpha_t = e^{-\int_0^t r_s ds}.$$

As observed above, pricing of claims requires the martingale probability measure Q under which the process $Z_t = \frac{S_t}{\alpha_t}$ is a martingale. The change of measure is done by using Ito's formula: pricing equation become

$$dS_t = S_t r_t dt + \sigma(S_t)dW_t,$$

where now W is a Q-Wiener process. The most important diffusion model is certainly the *Black-Schoels model* where $a(S_t) = aS_t$, $\sigma(S_t) = \sigma S_t$ and moreover the interest rate is assumed to be constant so that $\alpha_t = e^{-rt}$. The general stock price process is

$$dS_t = aS_t dt + \sigma S_t dW_t,$$

of more explicitly

$$\frac{dS_t}{S_t} = adt + \sigma dW_t,$$

where $dR_t = \frac{dS_t}{S_t}$ are the returns on stock process. The change of measure from P to Q yields the following pricing equation

$$dS_t = rS_t dt + \sigma S_t dW_t,$$

whose solution is $S_t = S_0 e^{(r-\frac{\sigma^2}{2})t-\sigma W_t}$. The price of a claim X at time T will be

$$C_t = e^{-r(T-t)}E^Q(X|\mathcal{F}_t).$$

If X is a call option $C_t = S_t\Phi(\delta) - Ke^{-r(T-t)}\Phi(\delta - \sigma\sqrt{T-t})$ where $\delta = \frac{log(S_t/K)+(r+\frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}}$, where $\Phi(x)$ is the distribution function of the $N(0,1)$ law, see Klebaner (1999).

Less intuitive is the pricing rule in the case where the model contains jumps as for example the following simple jump-diffusion model

$$S_t = S_0 e^{at+\sigma W_t + \sum_{k=1}^{N_t} Y_k},$$

where N_t is a Poisson process with intensity λ, Y_k are independent and identically distributed r.vs. denoting jump sizes. Here, we are only interested in emphasizing how important is the correct identification of the model for the strategy of pricing and hedging. For instance, in the univariate case, Cont (Cont, 2005) shows how the choice of two different models for a specified stock in such a way that both calibrates the same record of call options yields two prices of the same financial derivatives whose different is greater than 60 percent. When the model is bi-dimensional the situation can be worse and the identification of jumps become crucial.

1.4 The Realized Volatility and the Threshold Estimator

A crucial concept relative to the Financial Market movements is certainly the *market volatility* which gives a measure of risk and variability of prices. Traditionally, in the theory of financial economics the variation of asset prices is measured by looking at sums of outer products of returns calculated over small time periods. The mathematics of this is based on the *quadratic variation process* (see for example Protter, 1990). Asset pricing theory links the dynamics of increments of quadratic variation to the increments of risk premium. A wide econometric literature is available on this subject: Andersen, Bollerslev, Diebold and Labys (2001), Barndorff-Nielsen and Shephard (2003, 2004a, 2004b, 2004c), Comte and Renault (1998), Mancini (2004, 2005). Given a sample of n returns $r_j = X_{jh} - X_{(j-1)h} = \Delta_j X$, $j = 1, 2, ..., n$ observed at intervals h over the fixed period $[0, T]$ with $T = nh$ the *realized variance* or *realized volatility* is given by

$$RQV_T(X) = \sum_{j=1}^{n}(X_{jh} - X_{(j-1)h})^2 = \sum_{j=1}^{n}(\Delta_j X)^2.$$

We will show (chapter 2) that the process $(RQV_t(X))_{t\in[0,T]}$ converges in probability to the quadratic variation process of X, $([X]_t)_{t\in[0,T]}$. Typically, the quadratic variation contains either a diffusion part or a jump part: our purpose is to separate the contribution of each one. In particular, we are interested in a bivariate jump model $(X^{(1)}, X^{(2)})$, i.e., we study the *realized quadratic covariation*

$$RQC_T(X^{(1)}, X^{(2)}) = \sum_{j=1}^{n}(X^{(1)}_{jh} - X^{(1)}_{(j-1)h})(X^{(2)}_{jh} - X^{(2)}_{(j-1)h}) = \sum_{j=1}^{n}(\Delta_j X^{(1)})(\Delta_j X^{(2)}),$$

where we have two price (or log-price) processes X and Y. Obviously, the quadratic covariation process $([X^{(1)}, X^{(2)}]_t)_{t\in[0,T]}$ is the limit in probability of the process $(RQC_t(X^{(1)}, X^{(2)}))_{t\in[0,T]}$. Since our bivariate model contains jumps, we use an estimator that can identify the instants of jump on the basis of a discrete record of high frequency observations. In particular, we use a result due to Mancini (2005) who concentrates on the behaviour of squared increments of the process $(\Delta_j X^{(q)})^2 = (X^{(q)}_{t_j} - X^{(q)}_{t_{j-1}})^2$. The celebrate **Lévy modulus of continuity** tells us that the exact modulus of continuity of the paths of a Wiener process is $\sqrt{2hlog\frac{1}{h}}$, i.e., if $|t - s| < h$ then $|W_t - W_s| \le \sqrt{2hlog\frac{1}{h}}$, $P - a.s.$. In other words, the absolute value of the increments of the paths of W tends to zero as $\sqrt{2hlog\frac{1}{h}}$. This yields for small h, ω by ω, $\sup_{j=1,...,n} \frac{|\Delta_j(\sigma.W)|}{\sqrt{2hlog\frac{1}{h}}} \le M(\omega)$, $P - a.s.$, that is, the increments of the stochastic integral (continuous part of the model) have the same behaviour of the increments of the Wiener process. Then, when the activity of jumps is finite we can say that if, for small h, the square increments $(\Delta_j X^{(q)})^2 > r_h > \sqrt{2hlog\frac{1}{h}}$ we can think that it could not be generated by the continuous part of $X^{(q)}$ and thus *jumps have to be occurred*. The function of time interval r_h is a mapping such that $\lim_{h\to 0} r_h = 0$ and $\lim_{h\to 0} \frac{hlog\frac{1}{h}}{r_h} = 0$. Formally (Mancini, 2005), $1_{\{(\Delta_j X^{(q)})^2 \le r_h\}} = 1_{\{\Delta_j N^{(q)} = 0\}}$, $P - a.s.$, $\forall j = 1, 2, ..., n$, $q = 1, 2$. Hence, we are able to introduce a new estimator, the **threshold estimator**, defined by

$$\tilde{v}^{(n)}_{1,1}(X^{(1)}, X^{(2)})_T = \sum_{j=1}^{n} \Delta_j X^{(1)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} \Delta_j X^{(2)} 1_{\{(\Delta_j X^{(2)})^2 \le r_h\}}.$$

Even if the sum of cross products $\sum_{j=1}^{n} \Delta_j X^{(1)} \Delta_j X^{(2)}$ converges in probability to the global quadratic covariation $[X^{(1)}, X^{(2)}]_T = [X^{(1)}, X^{(2)}]^{(c)}_T + [X^{(1)}, X^{(2)}]^{(J)}_T$, this new estimator exploits the truncation principle to exclude the cross products between increments that are too much (properly) large, in such a way that the second term of the quadratic covariation, $[X^{(1)}, X^{(2)}]^{(J)}_T$, is eliminated. This fact is crucial for the portfolio hedging strategies. In fact, financial traders are used to estimate the diffusion part of the covariation by the sum of the cross products of increments,

$RQC_T(X^{(1)}, X^{(2)})$, which is an unbiased estimator of such a covariation if a jump component is included in the model, because it converges to the sum of the contributes, $[X^{(1)}, X^{(2)}]_T^{(c)} + [X^{(1)}, X^{(2)}]_T^{(J)}$.

1.5 Outline

Chapter 2 provides a more detailed overview of the theory of Stochastic processes and Stochastic Integrals. Moreover, it contains a brief introduction on the Lévy processes and their main properties. Chapter 3 focuses on the preliminary results in the case where the stock price dynamics is of the diffusion type. Chapter 4 has been written with Prof. Cecilia Mancini and it shows the main results of this work: in particular, we discuss the asymptotic behaviour of the *threshold estimator* both in the case of finite activity and in the case of infinite activity. In Chapter 5 we develop a simulation study to evaluate the performance of our estimator in small samples.

Chapter 2

Elements of general theory of stochastic processes

"I' son Beatrice che ti faccio andare;
vegno del loco ove tornar disio;
amor mi mosse che mi fa parlare."
(D. Alighieri, Divina Commedia, Inferno, Canto II)

2.1 Stochastic processes and Martingales

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a **filtered probability space**, where Ω is a set, $\mathcal{F}$ is a σ-field of subsets of Ω, P is a probability measure on $(\Omega, \mathcal{F})$ and $(\mathcal{F}_t)_{t\geq 0}$ is a **filtration**, i.e., an increasing family of sub-σ-fields, $\mathcal{F}_s \subset \mathcal{F}_t$, for each $s \leq t$. Define $\mathcal{F}_{t+} = \bigcap_{\epsilon>0} \mathcal{F}_{t+\epsilon}$. If $\mathcal{F}_{t+} = \mathcal{F}_t$ for each $t \geq 0$, we say that the family $(\mathcal{F}_t)_{t\geq 0}$ is *right-continuous*. Moreover, we consider filtrations which are complete, that is, if $\mathcal{N} = \{A \in \mathcal{F} : P(A) = 0\}$ we assume that $\mathcal{N} \subset \mathcal{F}_t$ for each t.

Definition 2.1.1. *Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ be a filtered probability space. A **stochastic process** is a mapping X from $\Omega \times \mathbb{R}_+$ to $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ such that for each $t \geq 0$ $X_t(\cdot) : \omega \mapsto X_t(\omega)$ is $\mathcal{F}_t$-measurable; in other words $X = (X_t(\omega))_{t\geq 0}$ is a family of r.v.s indexed by t. Moreover, the mappings $X_.(\omega) : t \mapsto X_t(\omega)$ are called* sample paths *or* paths *of X.*

To simplify the notation we write $X = (X_t)_{t\geq 0}$.

Definition 2.1.2. *A stochastic process $X = (X_t)_{t\geq 0}$ is said to be* continuous *if all sample paths are continuous, $P - a.s.$, that is, if there exists a set $\Omega^* \in \mathcal{F}$ with $P(\Omega^*) = 0$ such that for each $\omega \notin \Omega^*$ the function $t \mapsto X_t(\omega)$ is continuous. In the same way, we may define right-continuous with left limit (***cadlag***) processes and left-continuous with right-limit (***caglad***) processes.*

Definition 2.1.3. *Let $X = (X_t)_{t\geq 0}$ and $Y = (Y_t)_{t\geq 0}$ be two stochastic processes.*

1. *We say that X is a **modification** of Y if*

$$P(X_t = Y_t) = P\{\omega \in \Omega : X_t(\omega) = Y_t(\omega)\} = 1, \quad t \geq 0;$$

2. *we say that the processes X and Y are **indistinguishable** if*

$$P(X_t = Y_t, t \geq 0) = 1,$$

provided that $(X_t = Y_t, t \geq 0) = \bigcap_{t\geq 0}(X_t = Y_t) \in \mathcal{F}$.

Generally, (2) is strictly stronger than (1). However, in the case where X and Y have continuous sample paths the two conditions are equivalent.

Definition 2.1.4. *We say that the stochastic process $X = (X_t)_{t\geq 0}$ is **measurable** if it is measurable w.r.t. $\mathcal{B}_{\mathbb{R}_+} \otimes \mathcal{F}$, it is **adapted** to the filtration $(\mathcal{F}_t)_{t\geq 0}$ if for each $t \geq 0$ the r.v. X_t is $\mathcal{F}_t$-measurable and it is **progressively measurable** if for each $t \geq 0$ the mapping X restricted to $[0, t] \cap \mathbb{R}_+$ is $\mathcal{B}_{[0,t]} \otimes \mathcal{F}$-measurable, where $\mathcal{B}_{[0,t]}$ is the Borel σ-field of subsets of $[0, t]$.*

It is clear that each progressively measurable process is adapted. Moreover, we have the following

Proposition 2.1.5. *Each adapted and right-continuous process is progressively measurable.*

***Proof*.** We give the outline of the proof. Let $X = (X_t)_{t\geq 0}$ be a right-continuous process adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$ and define a sequence of right-continuous processes $\{X^{(n)}, n \in \mathbb{N}\} = \{(X_s^{(n)})_{s\geq 0}, n \in \mathbb{N}\}$ by

$$X_s^{(n)} = X_{\frac{k+1}{2^n}s}, \quad s \in [\frac{k}{2^n}t, \frac{k+1}{2^n}t[,$$

and

$$X_s^{(n)} = X_t, \quad s = t,$$

for some fixed $t \geq 0$. We can prove that $X_s^{(n)} \to X_s$ as $n \to \infty$ for every $s \leq t$. Then, we only have to show that $X^{(n)}$ is progressively measurable for each n. In fact, if $A \in \mathcal{B}_{\mathbb{R}}$ we have

$$\{(s, \omega) \in \mathbb{R}_+ \times \Omega : s \leq t, X_s(\omega) \in A\} =$$

$$\bigcup_{k=1}^{2^n-1} ([\frac{k}{2^n}t, \frac{k+1}{2^n}t[\times X_{\frac{k+1}{2^n}t}(\omega) \in A) \cup (\{t\} \times X_t \in A) \in \mathcal{B}_{[0,t]} \otimes \mathcal{F}_t,$$

so X is progressively measurable being a limit of progressively measurable mappings.

•

Definition 2.1.6. *A stochastic process $X = (X_t)_{t\geq 0}$ is said to be **predictable** if it is measurable with respect to the **predictable σ-field**, $\mathcal{P}$, which is the σ-field generated by all* caglad *processes defined on $\mathbb{R}_+ \times \Omega$*

$$\mathcal{P} = \sigma(H : H : \mathbb{R}_+ \times \Omega \to \mathbb{R}, \ H \text{ is caglad}).$$

Proposition 2.1.7. *Each adapted and* caglad *process is predictable.*

•

We omit the proof since it is similar to that of proposition 2.1.1.

Definition 2.1.8. *A nonnegative random variable $\tau : \Omega \to \mathbb{R}_+$ is a **stopping time** if $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$. The* stopping time σ-field, *denoted by $\mathcal{F}_\tau$, is defined to be*

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap (\tau \leq t) \in \mathcal{F}_t, \ \forall t \geq 0\}.$$

Definition 2.1.9. *An adapted stochastic process $X = (X_t)_{t\geq 0}$ is called **martingale** w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$ if*

1. *X_t is an integrable r.v. for every $t \geq 0$, i.e., $\sup_{t\geq 0} E|X_t| < \infty$;*
2. *$E(X_t|\mathcal{F}_s) = X_s$, $P-a.s.$, for each $s \leq t$.*

We often denote martingales by $X = (X_t, \mathcal{F}_t)_{t\geq 0}$ to clarify the filtration w.r.t. which X_t is a martingale.

Proposition 2.1.10. *Let X be a martingale. Then, there exists a unique modification Y of X which is* cadlag.

•

Definition 2.1.11. *Let X be a martingale such that $\sup_{t\geq 0} E|X_t|^2 < \infty$. Define by $[X]_t$ the unique adapted continuous increasing process such that $(X_t^2 - [X]_t, \mathcal{F}_t,\ t \geq 0)$ is a martingale.*

Lemma 2.1.12. *Let X be a martingale such that $\sup_{t\geq 0} E|X_t|^2 < \infty$. Then, $s < t \leq u < v$ we have $E[(X_t - X_s)(X_v - X_u)] = 0$, $P - a.s.$, and moreover $E[(X_t - X_s)^2|\mathcal{F}_s] = E[[X]_t - [X]_s|\mathcal{F}_s]$.*

***Proof*.** We can write

$$E[(X_t - X_s)(X_v - X_u)] = E\{E[(X_t - X_s)(X_v - X_u)|\mathcal{F}_u]\} =$$

$$E\{(X_t - X_s)E[(X_v - X_u)|\mathcal{F}_u]\} = E[(X_t - X_s)(X_u - X_u)] = 0.$$

Moreover,

$$E[(X_t - X_s)^2|\mathcal{F}_s] = E(X_t^2|\mathcal{F}_s) - 2E(X_tX_s|\mathcal{F}_s) + E(X_s^2|\mathcal{F}_s) =$$

$$E(X_t^2|\mathcal{F}_s) - 2X_sE(X_t|\mathcal{F}_s) + E(X_s^2|\mathcal{F}_s) = E(X_t^2|\mathcal{F}_s) - 2X_s^2 + E(X_s^2|\mathcal{F}_s) =$$

$$E(X_t^2|\mathcal{F}_s) - 2E(X_s^2|\mathcal{F}_s) + E(X_s^2|\mathcal{F}_s) = E(X_t^2 - X_s^2|\mathcal{F}_s),$$

$P - a.s.$. Now, since $X_t^2 - [X]_t$ is a martingale, we can write $P - a.s.$

$$0 = E\{[(X_t^2 - [X]_t)(X_s^2 - [X]_s)]|\mathcal{F}_s\} = E(X_t^2 - X_s^2|\mathcal{F}_s) - E([X]_t - [X]_s|\mathcal{F}_s) =$$

$$E[(X_t - X_s)^2|\mathcal{F}_s] - E([X]_t - [X]_s|\mathcal{F}_s),$$

as required.

•

Theorem 2.1.13. *(*Doob's sampling theorem*) Let X be a right-continuous martingale and let τ and ς be two stopping times such that $\tau \leq \varsigma$, $P - a.s.$. Then $E(X_\tau|\mathcal{F}_\varsigma) = X_\varsigma$, $P - a.s.$.*

•

We conclude this section with the definition of the Wiener process which is the most important stochastic process.

Definition 2.1.14. *An adapted stochastic process $W = (W_t)_{t\geq 0}$ with values in $\mathbb{R}$ is a **Wiener process** if*

1. *for any $s < t$, $W_t - W_s$ is independent of $\mathcal{F}_s$ (Independent increments);*
2. *for any $s \leq t$, the distribution of $W_t - W_s$ is $N(0, t - s)$;*
3. *$W_0 = 0$, $P - a.s.$.*

Proposition 2.1.15. *Let W be a Wiener process. Then, there exists a unique modification Y of W which is continuous.*

•

Proposition 2.1.16. *Let W be a Wiener process. Then, $W_t^2 - t$ is a martingale.*

***Proof*.** We have to prove that $E[(W_t^2 - t) - (W_s^2 - s)|\mathcal{F}_s] = 0$, $P - a.s.$, for every $s \leq t$. In fact, $E(W_t^2 - W_s^2|\mathcal{F}_s) = E[(W_t - W_s)^2|\mathcal{F}_s]$, and by independence of increments $E[(W_t - W_s)^2|\mathcal{F}_s] = E(W_t - W_s)^2 = t - s$; thus

$$E[(W_t^2 - t) - (W_s^2 - s)|\mathcal{F}_s] = E[(W_t - W_s)^2|\mathcal{F}_s] - E[t - s|\mathcal{F}_s] = (t - s) - (t - s) = 0, \quad P - a.s..$$

•

A simple consequence of this proposition is that by definition $[W]_t = t,\ P-a.s.$.

Proposition 2.1.17. *Let W be a Wiener process and let $\pi_n = \{t_{0,n}, t_{1,n}, ..., t_{j_n,n}\}$ be a sequence of partitions of the compact interval $[0,t]$ such that $0 = t_{0,n} < t_{1,n} < .... < t_{j_n,n} = t$ and $\max_{j=0,1,...,j_n-1} |t_{j+1,n} - t_{j,n}| \to 0$ as $n \to \infty$. Then*

$$\sum_{j=1}^{j_n-1} (W_{t_{j+1,n}} - W_{t_{j,n}})^2 \xrightarrow{\mathcal{L}^2} t.$$

Moreover, the sample paths of W, $W_\cdot(\omega) : t \mapsto W_t(\omega)$, are of unbounded variation in each compact interval of $\mathbb{R}_+$.

Proof. We have to prove that $E(\sum_{j=1}^{j_n-1}(W_{t_{j+1,n}} - W_{t_{j,n}})^2 - t)^2 \to 0,\ n \to \infty$. We can write

$$E[\sum_{j=1}^{j_n-1} (W_{t_{j+1,n}} - W_{t_{j,n}})^2 - t]^2 = E\{\sum_{j=1}^{j_n-1} [(W_{t_{j+1,n}} - W_{t_{j,n}})^2 - (t_{j+1,n} - t_{j,n})]\}^2.$$

By properties of the Wiener process, the set $((W_{t_{j+1,n}} - W_{t_{j,n}})^2 - (t_{j+1,n} - t_{j,n}))_{j=0,1,...,j_n-1}$ is a family of zero-mean Gaussian independent r.v.s, hence the expectation of cross products in the previous sum is zero

$$E\{[(W_{t_{j+1,n}} - X_{t_{j,n}})^2 - (t_{j+1,n} - t_{j,n})][(X_{t_{i+1,n}} - X_{t_{i,n}})^2 - (t_{i+1,n} - t_{i,n})]\} = 0,$$

$j, i = 0, 1, ..., j_n - 1,\ j \neq i$, then

$$E[\sum_{j=1}^{j_n-1} (W_{t_{j+1,n}} - W_{t_{j,n}})^2 - t]^2 = \sum_{j=1}^{j_n-1} E\{[(W_{t_{j+1,n}} - W_{t_{j,n}})^2 - (t_{j+1,n} - t_{j,n})]^2\} =$$

$$\sum_{j=1}^{j_n-1} (t_{j+1,n} - t_{j,n})^2 E[\frac{(W_{t_{j+1,n}} - W_{t_{j,n}})^2}{(t_{j+1,n} - t_{j,n})} - 1]^2.$$

The r.v. $\frac{(W_{t_{j+1,n}} - W_{t_{j,n}})^2}{(t_{j+1,n} - t_{j,n})} = (\frac{W_{t_{j+1,n}} - X_{t_{j,n}}}{\sqrt{(t_{j+1,n} - t_{j,n})}})^2$ has a Chi-square distribution with parameter 1. Therefore, the expectation $E[\frac{(W_{t_{j+1,n}} - W_{t_{j,n}})^2}{(t_{j+1,n} - t_{j,n})} - 1]^2$ is a quantity independent of j, say C, thus

$$E[\sum_{j=1}^{j_n-1} (W_{t_{j+1,n}} - W_{t_{j,n}})^2 - t]^2 = C \sum_{j=1}^{j_n-1} (t_{j+1,n} - t_{j,n})^2.$$

So

$$E[\sum_{j=1}^{j_n-1} (W_{t_{j+1,n}} - W_{t_{j,n}})^2 - t]^2 = C \sum_{j=1}^{j_n-1} (t_{j+1,n} - t_{j,n})^2 \leq$$

$$C \max_{j=0,1,...,j_n-1} |t_{j+1,n} - t_{j,n}| \sum_{j=1}^{j_n-1} |t_{j+1,n} - t_{j,n}| = Ct \max_{j=0,1,...,j_n-1} |t_{j+1,n} - t_{j,n}| \to 0,$$

as required. Now, we show that W is of unbounded variation on each $[a,b]$, $P-a.s.$, i.e., there exists a measurable set $\Omega_{[a,b]}$ with $P(\Omega_{[a,b]}) = 0$ such that for each $\omega \notin \Omega_{[a,b]}$ the sample paths $W_\cdot(\omega)$ is of unbounded variation. By the first part of the proof we can write that if $\pi_n = \{t_{0,n}, t_{1,n}, ..., t_{j_n,n}\}$ is a sequence of partitions of $[a,b]$, then $\sum_{j=1}^{j_n-1}(W_{t_{j+1,n}} - W_{t_{j,n}})^2 \xrightarrow{\mathcal{L}^2} b - a$. Moreover

$$\sum_{j=1}^{j_n-1} (W_{t_{j+1,n}} - W_{t_{j,n}})^2 \leq \max_{j=0,1,...,j_n-1} |W_{t_{j+1,n}} - W_{t_{j,n}}| \sum_{j=1}^{j_n-1} |W_{t_{j+1,n}} - W_{t_{j,n}}|.$$

Since the Wiener process has continuous sample paths, $\max_{j=0,1,\ldots,j_n-1}|W_{t_{j+1,n}} - W_{t_{j,n}}| \to 0$ as $n \to \infty$. If, by absurd, there exists a measurable set $\Omega^* \in \mathcal{F}$ with $P(\Omega^{'}) > 0$ such that for $\omega \in \Omega^*$, $\sum_{j=1}^{j_n-1}|W_{t_{j+1,n}} - W_{t_{j,n}}| \leq C(\omega) < \infty$ we would be as $n \to \infty$

$$\sum_{j=1}^{j_n-1}(W_{t_{j+1,n}} - W_{t_{j,n}})^2 \to 0, \quad \omega \in \Omega^{'},$$

which is a contradiction with the first statement of the proposition. Consider now all the compact intervals $[a,b]$ with $a,b \in \mathbb{Q}$. Let $\Upsilon = \bigcup_{a,b\in\mathbb{Q}:a\leq b}\Omega_{[a,b]}$. Obviously, $\Upsilon \in \mathcal{F}$ and $P(\Upsilon) \leq \sum_{a,b\in\mathbb{Q}:a\leq b}P(\Omega_{[a,b]}) = 0$. Hence, if $\omega \notin \Upsilon$ the sample paths $W_.(\omega)$ is of unbounded variation on each $[c,d] \subset \mathbb{R}_+$.

•

Finally, we recall a fundamental result on the sample paths of the Wiener process due to P. Lévy.

Theorem 2.1.18. ***(Lévy modulus of continuity)*** *Let W be a Wiener process. Then for every $T > 0$ we have*

$$\lim_{h\to 0}\sup_{s,t\in[0,T]:|t-s|\leq h}\frac{|W_t - W_s|}{\sqrt{2hlog\frac{1}{h}}} = 1, \quad P-a.s.$$

•

2.2 Stochastic Integrals

2.2.1 Stochastic integral with respect to a continuous square integrable martingale

We give a precise definition of stochastic integral w.r.t. a continuous square integrable martingale (i.e. the Wiener process). Let $\mathcal{M}_c^2$ be the class of all square integrable martingales defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$. We assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfies the usual conditions. We write $Z = (Z_t, \mathcal{F}_t)_{t\geq 0}$ an element of $\mathcal{M}_c^2$ and suppose $Z_0 = 0$, $P-a.s.$. Moreover, if we define $\|Z\|_t^{\mathcal{M}_c^2} = \sqrt{EZ_t^2}$ for all $Z \in \mathcal{M}_c^2$, we can endow it by the norm $\|Z\|^{\mathcal{M}_c^2} := \sum_n 2^{-n}(1 \wedge \|Z\|_n^{\mathcal{M}_c^2})$. The key point is to define precisely the class of integrand processes, because the integrator process has paths of unbounded variation and hence the pathwise Lebesgue-Stieltjes definition is not correct. We choice to define the stochastic integral for integrands which are progressively measurable processes, following Karatzas-Shreve (1991).

We call $\mathcal{H}$ the class of all progressively measurable processes satisfying the condition $\int_0^T Z_t^2 d[Z]_t < \infty$, for all $T > 0$, and we define a metric on it by

$$\|Z\|^{\mathcal{H}} := \sum_n 2^{-n}(1 \wedge \|Z\|_n^{\mathcal{H}}), \quad Z \in \mathcal{H} \tag{2.1}$$

where $\|Z\|_n^{\mathcal{H}} := \sqrt{\int_0^n Z_t^2 d[Z]_t}$.

Remark 2.2.1. *Recall that $[Z]_t$ is the unique adapted continuous increasing process such that $(Z_t^2 - [Z]_t, \mathcal{F}_t,\ t \geq 0)$ is a martingale. Hence, an integral w.r.t. it is well defined in the Lebesgue-Stieltjes sense. We call $[Z]_t$* ***quadratic variation process****. In the following we will speak of it with more details.*

We proceed with the definition of the stochastic integral and the presentation of its main properties for a particular class of integrands, the simple processes, from which by approximation we can reach to the definition for progressively measurable ones.

Definition 2.2.2. *A stochastic process $Y = (Y_t)_{t\geq 0}$ is said to be* ***simple*** *if it has the form*

$$Y_t(\omega) := \xi_0(\omega)1_{\{0\}}(t) + \sum_{j=0}^{\infty} \xi_j(\omega)1_{]t_j,t_{j+1}]}(t), \quad t \geq 0,\ \omega \in \Omega,$$

where $\{t_n, n \in \mathbb{N}\}$ is a sequence of real of real numbers such that $t_0 = 0$, $t_n < t_{n+1}$ for all n and $t_n \to \infty$ as $n \to \infty$, $\{\xi_n, n \in \mathbb{N}\}$ is a sequence of r.v.s such that ξ_n is $\mathcal{F}_{t_n}$-measurable and $\sup_{n\in\mathbb{N}} |\xi_n(\omega)| < \infty$, $\forall\omega \in \Omega$. We denote the class of all simple processes by $\mathcal{S}$ and we equip it by the same norm of $\mathcal{H}$, $\|\cdot\|^{\mathcal{H}}$.

Let $Z = (Z_t, \mathcal{F}_t)_{t\geq 0}$ be a continuous square integrable martingale, i.e. $Z \in \mathcal{M}_c^2$, and $Y \in \mathcal{S}$. The stochastic integral of Y with respect to Z is defined by

$$(Y.Z)_t = \int_0^t Y_s dZ_s := \sum_{j=0}^{\infty} \xi_j(Z_{t_{j+1}\wedge t} - Z_{t_j\wedge t}) = \sum_{j=0}^{n-1} \xi_j(Z_{t_{j+1}} - Z_{t_j}) + \xi_n(Z_t - Z_{t_n}), \quad t \geq 0.$$

provided $t_n < t \leq t_{n+1}$ for some finite n. It's not hard to derive the main properties of $(Y.Z)_t$.

1. $(Y.Z)_t$ is a continuous martingale w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$, i.e., $\forall s \leq t$

$$E[(Y.Z)_t|\mathcal{F}_s] = (Y.Z)_s,\ P-a.s.$$

Proof. We have to prove that $E[(Y.Z)_t - (Y.Z)_s|\mathcal{F}_s] = 0,\ P-a.s.$ Suppose the maximum generality, that is, suppose that s and t are in different subintervals $0 = t_0 < t_1 < \cdots < t_k < s < t_{k+1} < \cdots < t_n < t < t_{n+1} < \cdots$. Write

$$(Y.Z)_t = \sum_{j=0}^{n-1} \xi_j(Z_{t_{j+1}} - Z_{t_j}) + \xi_n(Z_t - Z_{t_n}) =$$

$$\sum_{j=0}^{k-1} \xi_j(Z_{t_{j+1}} - Z_{t_j}) + \xi_k(Z_{t_{k+1}} - Z_{t_k}) + \sum_{j=k+1}^{n-1} \xi_j(Z_{t_{j+1}} - Z_{t_j}) + \xi_n(Z_t - Z_{t_n}),$$

and

$$(Y.Z)_s := \sum_{j=0}^{k-1} \xi_j(Z_{t_{j+1}} - Z_{t_j}) + \xi_k(Z_s - Z_{t_k}).$$

Hence

$$E((Y.Z)_t - (Y.Z)_s|\mathcal{F}_s) = E(\xi_k(Z_{t_{k+1}} - Z_s)|\mathcal{F}_s) + \sum_{j=k+1}^{n-1} E(\xi_j(Z_{t_{j+1}} - Z_{t_j})|\mathcal{F}_s) + E(\xi_n(Z_t - Z_{t_n})|\mathcal{F}_s)$$

Consider the first term. Since ξ_k is $\mathcal{F}_s$-measurable (because $\mathcal{F}_{t_k} \subset \mathcal{F}_s$) we have

$$E(\xi_k(Z_{t_{k+1}} - Z_s)|\mathcal{F}_s) = \xi_k E(Z_{t_{k+1}} - Z_s)|\mathcal{F}_s) = \xi_k(Z_s - Z_s) = 0,\ P-a.s.$$

The second term is zero by using the law of iterated expectations and the fact that $\mathcal{F}_{t_j} \supset \mathcal{F}_s$, for $j = k+1, ..., n-1$

$$\sum_{j=k+1}^{n-1} E(\xi_j(Z_{t_{j+1}} - Z_{t_j})|\mathcal{F}_s) = \sum_{j=k+1}^{n-1} E\{E[\xi_j(Z_{t_{j+1}} - Z_{t_j})|\mathcal{F}_{t_j}]|\mathcal{F}_s\} =$$

$$\sum_{j=k+1}^{n-1} E\{\xi_j E[(Z_{t_{j+1}} - Z_{t_j})|\mathcal{F}_{t_j}]\mathcal{F}_s\} = 0,$$

$P-a.s.$. The third term is also zero

$$E(\xi_n(Z_t - Z_{t_n})|\mathcal{F}_s) = E\{E[\xi_n(Z_t - Z_{t_n})|\mathcal{F}_{t_n}]|\mathcal{F}_s\} = E\{H_n E[(Z_t - Z_{t_n})|\mathcal{F}_{t_n}]|\mathcal{F}_s\} = 0,\ P-a.s.$$

This concludes the proof.

We can also observe that $(Y.Z)_0 = 0$, $P-a.s.$, and $E(Y.Z)_t = 0$, for all t.

•

2. $E\{[(Y.Z)_t - (Y.Z)_s]^2|\mathcal{F}_s\} = E(\int_s^t Y_v^2 d[Z]_v|\mathcal{F}_s),\ P-a.s.$
Proof. Thanks to lemma 2.1.1, we can easily prove the statement 2. In fact, as in the proof of the property 1

$$E\{[(Y.Z)_t - (Y.Z)_s]^2|\mathcal{F}_s\} = E\{[\xi_k(Z_{t_{k+1}} - Z_s) + \sum_{j=k+1}^{n-1} \xi_j(Z_{t_{j+1}} - Z_{t_j}) + \xi_n(Z_t - Z_{t_n})]^2|\mathcal{F}_s\} =$$

$$E[\xi_k^2(Z_{t_{k+1}} - Z_s)^2|\mathcal{F}_s] + \sum_{j=k+1}^{n-1} E[\xi_j^2(Z_{t_{j+1}} - Z_{t_j})^2|\mathcal{F}_s] + E[\xi_n^2(Z_t - Z_{t_n})^2|\mathcal{F}_s].$$

However, by previous results

$$E[\xi_k^2(Z_{t_{k+1}} - Z_s)^2|\mathcal{F}_s] = E\{E[\xi_k^2(Z_{t_{k+1}} - Z_s)^2|\mathcal{F}_{t_k}]|\mathcal{F}_s\} =$$

$$E\{\xi_k^2 E[([Z]_{t_{k+1}} - [Z]_s)^2|\mathcal{F}_{t_k}]|\mathcal{F}_s\} = E\{E[\xi_k^2([Z]_{t_{k+1}} - [Z]_s)|\mathcal{F}_{t_k}]|\mathcal{F}_s\} = E[\xi_k^2([Z]_{t_{k+1}} - [Z]_s)|\mathcal{F}_s],$$

$P-a.s.$, while

$$\sum_{j=k+1}^{n-1} E[\xi_j^2(Z_{t_{j+1}} - Z_{t_j})^2|\mathcal{F}_s] = \sum_{j=k+1}^{n-1} \xi_j^2 E([Z]_{t_{j+1}} - [Z]_{t_j}|\mathcal{F}_s) = \sum_{j=k+1}^{n-1} E(\xi_j^2([Z]_{t_{j+1}} - [Z]_{t_j})|\mathcal{F}_s)$$

$P-a.s.$, and analogously $E[\xi_n^2(Z_t - Z_{t_n})^2|\mathcal{F}_s] = E[\xi_n^2([Z]_t - [Z]_{t_n})|\mathcal{F}_s]$, $P-a.s.$, so that

$$E\{[(Y.Z)_t - (Y.Z)_s]^2|\mathcal{F}_s\} =$$

$$E[\xi_k^2([Z]_{t_{k+1}} - [Z]_s) + \sum_{j=k+1}^{n-1} \xi_j^2([Z]_{t_{j+1}} - [Z]_{t_j}) + \xi_n^2([Z]_t - [Z]_{t_n})|\mathcal{F}_s] :=$$

$$E[\int_s^t Y_v^2 d[Z]_v|\mathcal{F}_s], \quad P-a.s.$$

Property 2 implies that the stochastic integral process $(Y.Z) = ((Y.Z)_t)_{t\geq 0}$ is a square integrable (and continuous) martingale, i.e., $(Y.Z) \in \mathcal{M}_c^2$.

•

3. $E(Y.Z)_t^2 = E\int_0^t Y_v^2 d[Z]_v$.
Proof. By property 2, taking $s = 0$ and using the law of iterated expectations

$$E\{E[(Y.Z)_t^2|\mathcal{F}_0]\} = E\{E[\int_0^t Y_v^2 d[Z]_v|\mathcal{F}_0]\} = E\int_0^t Y_v^2 d[Z]_v.$$

•

4. $[(Y.Z)]_t = \int_0^t Y_v^2 d[Z]_v$.
Proof. By property 2, we see that $\int_s^t Y_v^2 d[Z]_v$ is the quantity such that $((Y.Z)_t - (Y.Z)_s)^2$ is a martingale, so that setting $s = 0$, $[(Y.Z)]_t = \int_0^t Y_v^2 d[Z]_v$.

•

5. The stochastic integral satisfies an **isometry** property, that is, $\|(Y.Z)\|^{\mathcal{M}_c^2} = \|Y\|^{\mathcal{H}}$.
   ***Proof*.** The statement is an immediate consequence of properties 3.

•

To extend the definition of stochastic integral from simple processes to progressively measurable processes we need a useful approximation result. The key lemma which follows can be found in Karatzas-Shreve (1991).

Lemma 2.2.3. *Let $Z = (Z_t, \mathcal{F}_t)_{t\geq 0}$ be a martingale and let $A = (A_t, \mathcal{F}_t)_{t\geq 0}$ be a continuous and increasing process adapted to $(\mathcal{F}_t)_{t\geq 0}$. If $Y = (Y_t, \mathcal{F}_t)_{t\geq 0}$ is a progressively measurable process such that $E\int_0^T Y_t^2 dA_t < \infty$ for all $T > 0$, then there exists a sequence of simple processes, $\{Y^{(n)}, n \in \mathbb{N}\} = \{(Y_t^{(n)})_{t\geq 0}, n \in \mathbb{N}\} \subset \mathcal{S}$ such that*

$$\sup_{T>0} E\int_0^T |Y_t^{(n)} - Y_t|^2 dA_t \to 0, \quad \textit{as } n \to \infty.$$

•

This lemma establishes a fundamental property of simple processes given by the following

Proposition 2.2.4. *The class $\mathcal{S}$ of simple processes is dense in $\mathcal{H}$ under the metric generated by the norm $\|\cdot\|^{\mathcal{H}}$.*

•

This means that for all $Y \in \mathcal{H}$ there exists a sequence $\{Y^{(n)}, n \in \mathbb{N}\} \subset \mathcal{S}$ such that $\|Y^{(n)} - Y\|^{\mathcal{H}} \to 0$ as $n \to \infty$. Therefore, by using isometry property, it's easy to prove that the sequence of stochastic integral $\{(Y^{(n)}.Z), n \in \mathbb{N}\}$ is a Cauchy sequence. We have by linearity of the integral

$$\|(Y^{(n)}.Z) - (Y^{(m)}.Z)\|^{\mathcal{M}_c^2} = \|((Y^{(n)} - Y^{(m)}).Z)\|^{\mathcal{M}_c^2} = \|Y^{(n)} - Y^{(m)}\|^{\mathcal{H}} \to 0, \quad \textit{as } n, m \to \infty.$$

So, there exists a process $((Y.Z)_t)_{t\geq 0} \in \mathcal{M}_2^c$ which is the limit of the sequence $\{(Y^{(n)}.Z), n \in \mathbb{N}\} = \{((Y^{(n)}.Z)_t)_{t\geq 0}, n \in \mathbb{N}\} \subset \mathcal{M}_2^c$, since it is proved that $\mathcal{M}_2^c$ is a complete metric space (more precisely it is an Hilbert space). In other words $\|(Y^{(n)}.Z) - (Y.Z)\|^{\mathcal{M}_c^2} \to 0$, as $n \to \infty$.

Remark 2.2.5. *The existence of a sequence of simple processes which converges to any stochastic processes Y can be better understood taking into account the following fact. Let ϑ_n be a sequence of functions defined on $\mathbb{R}_+$ by*

$$\vartheta_n(x) = \frac{[2^n x]}{2^n}, \quad n = 1, 2, ..., \quad x \in \mathbb{R}_+.$$

Such functions are simple. Moreover, it's easy to see that $0 \leq x - \vartheta_n(x) \leq 2^{-n}$ and hence $\vartheta_n(x) \to x$, $\forall x \in \mathbb{R}_+$. So, if Y is a progressively measurable process we can construct the sequence $\vartheta_n(Y)$ which converges to Y itself as desired.

Remark 2.2.6. *Since the metric defined on $\mathcal{M}_c^2$ is the mean square metric we can also write that $(Y^{(n)}.Z)_t \xrightarrow{\mathcal{L}_2} (Y.Z)_t$, $\forall t \geq 0$, taking into account that the process $(Y.Z)_t$ is a square integrable r.v. for a fixed t.*

This limit stochastic integral process $(Y.Z)$ obviously satisfies the properties 1-5 relative to simple integrands. In particular, since $(Y.Z) \in \mathcal{M}_c^2$, it is a square integrable martingale. Moreover, to see that property 2 is satisfied it suffices to observe the following fact. Let $\{(Y^{(n)}.Z)_t, n \in N\}$

and $\{(Y^{(n)}.Z)_s, n \in N\}$ be two sequence converging in L_2 to $(Y.Z)_t$ and $(Y.Z)_s$ respectively, for $s < t$. Now, if $B \in \mathcal{F}_s$

$$E\{1_B[(Y.Z)_t - (Y.Z)_s]^2\} =$$

$$\lim_n E\{1_B[(Y^{(n)}.Z)_t - (Y^{(n)}.Z)_s]^2\} = \lim_n E(E\{1_B[(Y^{(n)}.Z)_t - (Y^{(n)}.Z)_s]^2\}|\mathcal{F}_s) =$$

$$\lim_n E(1_B E\{[(Y^{(n)}.Z)_t - (Y^{(n)}.Z)_s]^2|\mathcal{F}_s\}) = \lim_n E(1_B E\{[(Y^{(n)}.Z)_t - (Y^{(n)}.Z)_s]^2|\mathcal{F}_s\}) =$$

$$\lim_n E(1_B E[\int_s^t (Y_v^{(n)})^2 d[Z]_v|\mathcal{F}_s]) = \lim_n E(E[1_B \int_s^t (Y_v^{(n)})^2 d[Z]_v|\mathcal{F}_s]) = \lim_n E(1_B \int_s^t (Y_v^{(n)})^2 d[Z]_v)$$

$$= E(1_B \int_s^t Y_v^2 d[Z]_v).$$

This proves that $(Y.Z)$ satisfies property 2 and consequently properties 3, 4 and 5. We conclude with the formal definition of the stochastic integral process.

Definition 2.2.7. *Let $Y \in \mathcal{H}$. The stochastic integral of Y w.r.t. $Z \in \mathcal{M}_c^2$ is the unique, square-integrable martingale $(Y.Z) = ((Y.Z)_t, \mathcal{F}_t)_{t\geq 0}$ which satisfies $\|(Y^{(n)}.Z) - (Y.Z)\|^{\mathcal{M}_c^2} \to 0$, as $n \to \infty$, for every sequence $\{Y^{(n)}, n \in \mathbb{N}\} \subset \mathcal{H}$ such that $\|Y^{(n)} - Y\|^{\mathcal{H}} \to 0$. We write*

$$(Y.Z)_t = \int_0^t Y_s dZ_s, \quad t \geq 0.$$

The most important example of stochastic integral w.r.t. a square integrable martingale is certainly the one where the integrator is the Wiener process $W = (W_t, \mathcal{F}_t)_{t\geq 0}$. Since, as it's easy to show, $[W]_t = t$, $P - a.s.$, we can write

$$E(Y.W)_t^2 = E\int_0^t Y_v^2 dv,$$

and

$$[(Y.W)]_t = \int_0^t Y_v^2 dv.$$

Moreover, when the integrand is a deterministic function, say $t \mapsto g(t) \equiv g_t$, the stochastic integral $(g.W)$ is a Gaussian process.

Definition 2.2.8. *A stochastic process $(X_t)_{t\geq 0}$ is said to be Gaussian if $\forall d \in \mathbb{N}$ and $\forall i_1, ..., i_d \in \mathbb{R}_+$, $\boldsymbol{X} = (X_{i_1}, ..., X_{i_d})$ is a d-dimensional Gaussian random variable.*

Proposition 2.2.9. *Let g be a deterministic function such that $\sup_{t\geq 0} \int_0^t g_s^2 ds < \infty$, then the process $(g.W) = ((g.W)_t)_{t\geq 0}$ is Gaussian.*

***Proof*.** Firstly, suppose that g is a deterministic simple function, i.e., $g_t = \lambda_0 1_{\{0\}}(t) + \sum_{j=0}^{\infty} \lambda_j 1_{]t_j,t_{j+1}]}(t)$ for $\lambda_0, \lambda_1, ... \in \mathbb{R}$; by definition

$$(g.W)_t = \sum_{j=0}^{n-1} \lambda_j (W_{t_{j+1}} - W_{t_j}) + \lambda_n (W_t - W_{t_n}), \quad t \geq 0.$$

Therefore, $\forall d \in \mathbb{N}$ and $\forall i_1, ..., i_d \in \mathbb{R}_+$ we can write

$$(g.W)_{i_k} = \sum_{j=0}^{n_k-1} \lambda_j (W_{t_{j+1}} - W_{t_j}) + \lambda_{n_k} (W_{i_k} - W_{t_{n_k}}), \quad k = 1, 2, ..., d.$$

The r.v. $(g.W)_{i_k}$ is Gaussian because is a linear combination of Gaussian r.v.s, $W_{t_{j+1}} - W_{t_j} \sim N(0, t_{j+1} - t_j)$ and $W_{i_k} - W_{t_{n_k}} \sim N(0, i_k - t_{n_k})$. Moreover, it has zero mean and variance given by

$$E[(g.W)_{i_k}]^2 = \sum_{j=0}^{n_k-1} \lambda_j^2 E(W_{t_{j+1}} - W_{t_j})^2 + \lambda_{n_k}^2 E(W_{i_k} - W_{t_{n_k}})^2 =$$

$$\sum_{j=0}^{n_k-1} \lambda_j^2(t_{j+1}-t_j)+\lambda_{n_k}^2(i_k-t_{n_k}),$$

since the Wiener process has independent increments. Now, remark that $(g.W)_t$ also has independent increments, i.e., $(g.W)_{i_k}-(g.W)_{i_{k-1}}$ for $k=1,...,d$, are mutually independent. This implies that the d-dimensional r.v. $((g.W)_{i_1},(g.W)_{i_2}-(g.W)_{i_1},....,(g.W)_{i_d}-(g.W)_{i_{d-1}})$ is multivariate Gaussian because its independent elements have marginal Gaussian laws. Then, since $((g.W)_{i_1},....,(g.W)_{i_d})$ can be obtained by a linear transformation of

$$((g.W)_{i_1},(g.W)_{i_2}-(g.W)_{i_1},....,(g.W)_{i_d}-(g.W)_{i_{d-1}}),$$

we see that it is a d-dimensional Gaussian r.v.. In fact, we may write

$$\begin{pmatrix} (g.W)_{i_1} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ (g.W)_{i_d} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & .. & .. & 0 \\ 0 & -1 & 1 & 0 & .. & 0 \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot \\ 0 & 0 & .. & .. & -1 & 1 \end{pmatrix}^{-1} \times \begin{pmatrix} (g.W)_{i_1} \\ (g.W)_{i_2}-(g.W)_{i_1} \\ \cdot \\ \cdot \\ \cdot \\ (g.W)_{i_d}-(g.W)_{i_{d-1}} \end{pmatrix}.$$

For general integrands g satisfying the condition of square integrability, we take a sequence of simple functions $\{g^{(n)}, n \in \mathbb{N}\}$ converging to g_t, for all t; by the properties of the stochastic integral we have

$$((g^{(n)}.W)_{i_1},....,(g^{(n)}.W)_{i_d}) \xrightarrow{\mathcal{L}_2} ((g.W)_{i_1},....,(g.W)_{i_d})$$

as $n \to \infty$. Since normality of laws preserves under $\mathcal{L}^2$-convergence, the statement is proved.

•

The definition of stochastic integral given in this paragraph is valid even when $\mathbb{R}_+$ is replaced by the compact interval $[0,T]$, for $T>0$, that is for progressively measurable processes of the form $Y=(Y_t)_{t\in[0,T]}$. In fact, it suffices considering the space $\mathcal{H}_{[0,T]}$ which is the class of processes $Y \in \mathcal{H}$ such that $Y_t(\omega)=0$, $P-a.s.$, for every $t>T$.

2.2.2 Stochastic integral with respect to a semimartingale

The preceding definition of stochastic integral is not general because it is restricted to continuous processes, that is, both the integrator and the stochastic integral process have continuous trajectories, which is often a stronger restriction in applications, where it's crucial to consider processes whose paths may have discontinuities. In particular, we are interested in integrator processes with possibly *cadalg* paths. More precisely, we speak of semimartingales.

Exactly as in the preceding subsection we first give the definition of stochastic integral w.r.t. a restricted class of processes.

Definition 2.2.10. *A stochastic process $H=(H_t)_{t\geq 0}$ is said to be **simple predictable** if it has the form*

$$H_t(\omega) := \zeta_0(\omega)1_{\{0\}}(t)+\sum_{j=0}^{n}\zeta_j(\omega)1_{]\tau_j,\tau_{j+1}]}(t), \quad t\geq 0,\ \omega\in\Omega,$$

where $\{\tau_n, n\in\mathbb{N}\}$ is a finite sequence of stopping times such that $\tau_0=0$, $P-a.s.$, $\tau_0\leq\tau_1\leq\cdots\leq\tau_n\leq\tau_{n+1}<\infty$, $P-a.s.$, ζ_j is $\mathcal{F}_{\tau_j}$-measurable and $\sup_j|\zeta_j|<\infty$, $P-a.s.$. We denote the class of all simple predictable processes by $\mathcal{S}_p$.

Remark that if the sequence of stopping times is such that $0=\tau_0\leq\tau_1\leq\cdots\leq\tau_n\leq\tau_{n+1}=T$, $P-a.s.$, for some $T>0$, we could restrict our discussion on the compact $[0,T]$, without significantly changes in what follows.

We endow $\mathcal{S}_p$ by the uniform convergence metric generated by the norm $\|H\|_{sup} := \sup_{(t,\omega)\in\mathbb{R}_+\times\Omega} |H_t(\omega)|$, $\forall H \in \mathcal{S}_p$.

Let $\mathcal{E}$ be the space of all finite-valued r.v.s topologized by the convergence in probability, i.e. $\|Z\|^{\mathcal{E}} := E\frac{|Z|}{1+|Z|}$, $\forall Z \in \mathcal{E}$. Then, given a stochastic process $(X_t, \mathcal{F}_t)_{t\geq 0}$, we can define a mapping $\langle H.X\rangle : \mathcal{S}_p \to \mathcal{E}$ by

$$\langle H.X\rangle = \zeta_0 X_0 + \sum_{j=0}^{n} \zeta_j (X_{\tau_{j+1}} - X_{\tau_j}).$$

Definition 2.2.11. *We say that the random process $(X_t, \mathcal{F}_t)_{t\geq 0}$ is a* ***semimartingale*** *if it is* cadlag*, adapted and if the mapping $\langle H.X\rangle$ is continuous $\forall H \in \mathcal{S}_p$; that is, if for every sequence of simple predictable processes $\{H^{(n)}, n \in \mathbb{N}\}$ converging to H, the sequence of r.v.s $\{\langle H^{(n)}.X\rangle, n \in \mathbb{N}\}$ converges in probability to $\langle H.X\rangle$; formally*

$$X \text{ is a semimartingale if } \|H^{(n)} - H\|_{sup} \to 0 \Rightarrow \|\langle H^{(n)}.X\rangle - \langle H.X\rangle\|^{\mathcal{E}} \to 0 \text{ as } n \to \infty.$$

For example, let's show that each square integrable martingale is a semimartingale. Let $(X_t, \mathcal{F}_t)_{t\geq 0}$ be a square integrable martingale. We take a sequence of simple processes, $H^{(n)}$, such that $\|H^{(n)} - H\|_{sup} \to 0$. What we have to prove is that $\|\langle H^{(n)}.X\rangle - \langle H.X\rangle\|^{\mathcal{E}} \to 0$. By linearity of the mapping $\langle\cdot,\cdot\rangle$ the problem is to verify the convergence to zero of $\|\langle (H^{(n)} - H).X\rangle\|^{\mathcal{E}}$, which is established if $E|\langle (H^{(n)} - H).X\rangle|^2 \to 0$. Suppose without loss of generality that $X_0 = 0$, $P-a.s.$. Then

$$E|\langle (H^{(n)} - H).X\rangle|^2 = E[\sum_{j=0}^{n} (\zeta_j^{(n)} - \zeta_j)(X_{\tau_{j+1}} - X_{\tau_j})]^2 = E\sum_{j=0}^{n} (\zeta_j^{(n)} - \zeta_j)^2 (X_{\tau_{j+1}} - X_{\tau_j})^2,$$

while the expectation of cross products vanishes because the summands are increments of martingales. Now, we use the Doob's sampling theorem to obtain

$$E\sum_{j=0}^{n} (\zeta_j^{(n)} - \zeta_j)^2 (X_{\tau_{j+1}} - X_{\tau_j})^2 \leq \|H^{(n)} - H\|_{sup}^2 E\sum_{j=0}^{n} (X_{\tau_{j+1}} - X_{\tau_j})^2 =$$

$$\|H^{(n)} - H\|_{sup}^2 \sum_{j=0}^{n} E(X_{\tau_{j+1}}^2 + X_{\tau_j}^2 - 2E[E(X_{\tau_{j+1}} X_{\tau_j} | \mathcal{F}_{\tau_j})]) =$$

$$\|H^{(n)} - H\|_{sup}^2 \sum_{j=0}^{n} E(X_{\tau_{j+1}}^2 + X_{\tau_j}^2 - 2E[X_{\tau_j} E(X_{\tau_{j+1}} | \mathcal{F}_{\tau_j})])$$

$$\|H^{(n)} - H\|_{sup}^2 \sum_{j=0}^{n} E(X_{\tau_{j+1}}^2 + X_{\tau_j}^2 - 2X_{\tau_j}^2) = \|H^{(n)} - H\|_{sup}^2 \sum_{j=0}^{n} E(X_{\tau_{j+1}}^2 - X_{\tau_j}^2).$$

Thus

$$E|\langle (H^{(n)} - H).X\rangle|^2 \leq \|H^{(n)} - H\|_{sup}^2 \sum_{j=0}^{n} E(X_{\tau_{j+1}}^2 - X_{\tau_j}^2) = \|H^{(n)} - H\|_{sup}^2 E X_{\tau_{n+1}}^2 \to 0,$$

so that, $\langle (H^{(n)} - H).X\rangle$ converges to $\langle H.X\rangle$ in $\mathcal{L}^2$ and hence in probability.

The definition of stochastic integral requires a new type of metric which endows the space $\mathcal{S}_p$, the space of all adapted and *cadlag* processes, $\mathcal{D}$ and the space of all adapted and *caglad* processes, $\mathcal{G}$.

Definition 2.2.12. *We say that a sequence of processes $\{Y^{(n)}, n \in \mathbb{N}\}$ converges* ***uniformly on compacts in probability****, (ucp), if*

$$\|Y^{(n)} - Y\|_t^* := \sup_{s\leq t} |Y_s^{(n)} - Y_s| \xrightarrow{P} 0, \quad as\ n \to \infty, \quad \forall t \geq 0.$$

Moreover, a compatible metric with the *ucp* convergence is given by

$$\rho_{ucp}(Y,Z) := \sum_n 2^{-n} E(1 \wedge (Y-Z)^*_n),$$

for any couple of processes in $\mathcal{D}$ or in $\mathcal{G}$.

The following theorem (Protter, 1990) is crucial for the extension of the definition of stochastic integral to *caglad* integrands.

Theorem 2.2.13. *$\mathcal{S}_p$ is dense in $\mathcal{G}$ in* ucp *topology, that is, $\forall Y \in \mathcal{G}$ there exists a sequence $\{H^{(n)}, n \in \mathbb{N}\} \subset \mathcal{S}_p$ such that $\|H^{(n)} - Y\|^*_t \xrightarrow{P} 0$, $\forall t \geq 0$.*

•

Definition 2.2.14. *Let $H \in \mathcal{S}_p$ and let X be an adapted,* cadlag *process. The* ***stochastic integral*** *of H w.r.t. X, $(H.X) = ((H.X)_t)_{t\geq 0}$, is the mapping $(H.X) : (\mathcal{S}_p, \rho_{ucp}(\cdot,\cdot)) \to (\mathcal{D}, \rho_{ucp}(\cdot,\cdot))$ defined as*

$$(H.X)_t := \zeta_0 X_0 + \sum_{j=0}^{n} \zeta_j (X^{\tau_{j+1}} - X^{\tau_j}) = \zeta_0 X_0 + \sum_{j=0}^{n} \zeta_j (X_{\tau_{j+1}\wedge t} - X_{\tau_j \wedge t}), \quad t \geq 0.$$

Proposition 2.2.15. *If X is a semimartingale, then the mapping $(H.X)$ is continuous.*

•

Therefore, we are able to give the following

Definition 2.2.16. *Let $Y \in \mathcal{G}$ and X a semimartingale. The continuous (linear) mapping $(Y.X)_t$: $(\mathcal{G}, \rho_{ucp}(\cdot,\cdot)) \to (\mathcal{D}, \rho_{ucp}(\cdot,\cdot))$, obtained as extension of $(H.X)_t : (\mathcal{S}_p, \rho_{ucp}(\cdot,\cdot)) \to (\mathcal{D}, \rho_{ucp}(\cdot,\cdot))$ is called the* ***stochastic integral*** *of Y w.r.t. X. We write*

$$(Y.X)_t = \int_0^t Y_s dX_s, \quad t \geq 0,$$

Moreover, $\int_0^\infty Y_s dX_s := \lim_{t\to\infty} \int_0^t Y_s dX_s$ when the last integral exists.

Remark 2.2.17. *When the integrator process is a semimartingale, the integrands are represented by* caglad *processes; intuitively, this means that an observer approaching t can be predict the value of the process at t: jumps (discontinuities) are not sudden events.*

We present the main properties of the stochastic integral in form of theorem.

Theorem 2.2.18. *Let $Y \in \mathcal{G}$ and X be a semimartingale. Then*

1. *The jump process $\Delta(Y.X)_t$ is indistinguishable from $Y_t \Delta X_t$, i.e.,*

   $$P\{\omega \in \Omega : \text{ the mappings } t \mapsto \Delta(Y.X)_t(\omega) \text{ and } t \mapsto Y_t(\omega)\Delta X_t(\omega) \text{ are the same function}\} = 1;$$

2. *Let τ be a stopping time; then $(Y.X)^\tau = (Y.(X)^\tau)$;*

3. *The stochastic integral $(Y.X)$ is a semimartingale;*

4. *If $Z \in \mathcal{G}$ we have $(Z.(Y.X) = ((ZY).X)$.*

5. *Let $\tau_{0,n} \leq \tau_{1,n} \leq \cdots \leq \tau_{j_n,n}$ be a sequence of random partitions of $\mathbb{R}_+$ such that $\sup_j \tau_{j,n} \to \infty$, $P-a.s.$, and $\sup_j |\tau_{j+1,n} - \tau_{j,n}| \to 0$, $P-a.s.$. Suppose $Y \in \mathcal{G}$ or $Y \in \mathcal{D}$ then*

   $$\sum_{j=1}^{j_n} Y_{\tau_{j,n}} (X^{\tau_{j+1,n}} - X^{\tau_{j,n}}) \to ((Y_-).X), \quad \text{in ucp.}$$

•

Remark 2.2.19. *The process Y_- is defined by $Y_- = (Y_{t-})_{t\geq 0}$ where $Y_{t-} = \lim_{s\uparrow t} Y_s$. This means that if $Y \in \mathcal{G}$ then $Y_- \in \mathcal{G}$ while if $Y \in \mathcal{D}$ then $Y_- \in \mathcal{G}$.*

Theorem 2.2.20. *(Protter, 1990) If X is a semimartingale with sample paths of bounded variation on compacts, then $(Y.X)$ is indistinguishable from the Lebesgue-Stiltjes integral, computed path by path.*

•

2.3 Quadratic Variation and Quadratic Covariation

Let X be a semimartingale and we assume that $X_0 = 0$, $P - a.s.$. Consider a sequence of random partition as in theorem 2.2.2(5). We are interested in the uniform limit in probability of the sequence of processes $\{U^{(n)}(X), n \in \mathbb{N}\} = \{(U_t^{(n)}(X)_{t\geq 0}, n \in \mathbb{N}\}$ defined by

$$U^{(n)}(X)_t := \sum_{j=0}^{j_n-1} (X^{\tau_{j+1,n}} - X^{\tau_{j,n}})^2, \quad t \geq 0.$$

Since

$$U^{(n)}(X)_t = \sum_{j=0}^{j_n-1} (X^{\tau_{j+1,n}} - X^{t_{j,n}})^2 =$$

$$\sum_{j=0}^{j_n-1} [(X^{\tau_{j+1,n}})^2 - (X^{\tau_{j,n}})^2 - 2X^{\tau_{j,n}}(X^{\tau_{j+1,n}} - X^{\tau_{j,n}})] =$$

$$(X^{\tau_{n+1,n}})^2 - 2\sum_{j=0}^{j_n-1} X^{\tau_{j,n}}(X^{\tau_{j+1,n}} - X^{\tau_{j,n}}).$$

By using theorem 2.2.2(5) we get

$$U^{(n)}(X) \to [X], \quad \textit{in ucp},$$

that is $\sup_{s\leq t} |U^{(n)}(X)_t - [X]_s| \xrightarrow{P} 0$, or

$$U^{(n)}(X)_t \xrightarrow{P} X_t^2 - 2((X_-).X)_t = X_t^2 - 2\int_0^t X_{v-}dX_v = [X]_t,$$

uniformly in t. $[X] = ([X]_t)_{t\geq 0}$ is called **quadratic variation process** of X. We immediately see that it is an increasing (and adapted) process because if $s \leq t$, $U^{(n)}(X)_t$ contains more nonnegative terms than $U^{(n)}(X)_s$ and hence $U^{(n)}(X)_s \leq U^{(n)}(X)_t$. Since convergence in *ucp* preserves signs, we conclude that $[X]_s \leq [X]_t$, $P - a.s.$.

Remark 2.3.1. *When X is a square integrable martingale (progressively measurable) with continuous paths, the integral $\int_0^t X_v dX_v$ is also a martingale. That is, $[X]_t$ is exactly the adapted, increasing process defined in section 2.1.1 which guarantees that $X_t^2 - [X]_t$ is a martingale.*

Since the quadratic variation process has (by definition) *cadlag* paths, we have the following

Proposition 2.3.2. $\Delta[X]_t = (\Delta X_t)^2$.

Proof. Since $[X]_t = X_t^2 - 2((X_-).X)_t$ we have

$$\Delta[X]_t = \Delta(X_t^2 - 2((X_-).X)_t) = \Delta(X^2)_t - 2\Delta((X_-).X)_t = \Delta(X^2)_t - 2(X_{t-})\Delta X_t,$$

by properties of stochastic integral. Moreover

$$(\Delta X_t)^2 = (X_t - X_{t-})^2 = X_t^2 + X_{t-}^2 - 2X_t X_{t-} = X_t^2 - X_{t-}^2 - 2X_{t-}(X_t - X_{t-}) = \Delta(X^2)_t - 2(X_{t-})\Delta X_t,$$

so that $(\Delta X_t)^2 = \Delta[X]_t$, as required.

•

The quadratic variation process of a semimartingale can be decomposed into a continuous part and a purely discontinuous part, i.e.

$$[X]_t = [X]_t^{(c)} + [X]_t^{(d)},$$

where $[X]_t^{(d)} = \sum_{s\leq t}(\Delta X_s)^2$. We say that X is a **quadratic pure jump semimartingale** if and only if $[X]_t^{(c)} = 0$.

Now, suppose that the time horizon is fixed at $T > 0$, and let $\{0 = t_{0,n} < t_{1,n} < \cdots < t_{j_n,n} = T\}$ a sequence of non-random partitions of $[0,T]$, such that $\sup_{0\leq j\leq j_n-1}|t_{j+1,n} - t_{j,n}| \to 0$, as $n \to \infty$. Define **realized variance** of X on $[0,T]$ the quantity

$$u^{(n)}(X)_T := \sum_{j=0}^{j_n-1}(X_{t_{j+1,n}} - X_{t_{j,n}})^2,$$

while the **realized variance process** is given by the process $u^{(n)} = (u^{(n)}(X)_t)_{t\in[0,T]}$, for each n, where $u^{(n)}(X)_t = \sum_{j=0}^{j_n-1}(X_{t_{j+1,n}\wedge t} - X_{t_{j,n}\wedge t})^2$. Theorem 2.2.2(5) tell us that

$$u^{(n)}(X) \to [X], \quad \textit{in ucp},$$

or

$$\sup_{s\leq t}|u^{(n)}(X)_s - [X]_s| \xrightarrow{P} 0, \quad \textit{uniformly in } t.$$

By proposition 2.1.6 we see that $u^{(n)}(W)_t \xrightarrow{P} t$, if $W = (W_t)_{t\geq 0}$ is a Wiener process. Moreover, if $X_t = (\sigma.W)_t = \int_0^t \sigma_s dW_s$, for $t \in [0,T]$, as we have seen in subsection 0.1.1,

$$u^{(n)}((\sigma.W))_T \xrightarrow{P} [(\sigma.W)]_T = \int_0^T \sigma_s^2 ds,$$

If we consider two semimartingales X and Y with the usual assumption $X_0 = 0$ and $Y_0 = 0$, $P-a.s.$, we can introduce the sequence of processes $\{V^{(n)}(X,Y), n \in \mathbb{N}\} = \{(V^{(n)}(X,Y)_t)_{t\geq 0}, n \in \mathbb{N}\}$ defined by

$$V^{(n)}(X,Y)_t := \sum_{j=0}^{j_n-1}(X^{\tau_{j+1,n}} - X^{\tau_{j,n}})(Y^{\tau_{j+1,n}} - Y^{\tau_{j,n}}).$$

Since

$$V^{(n)}(X,Y)_t = \sum_{j=0}^{j_n-1}(X^{\tau_{j+1,n}} - X^{\tau_{j,n}})(Y^{\tau_{j+1,n}} - Y^{\tau_{j,n}}) =$$

$$\sum_{j=0}^{j_n-1}[X^{\tau_{j+1,n}}Y^{\tau_{j+1,n}} - X^{\tau_{j,n}}Y^{\tau_{j,n}} - X^{\tau_{j,n}}(Y^{\tau_{j+1,n}} - Y^{\tau_{j,n}}) - Y^{\tau_{j,n}}(X^{\tau_{j+1,n}} - X^{\tau_{j,n}})] =$$

$$X^{\tau_{n+1,n}}Y^{\tau_{n+1,n}} - \sum_{j=0}^{j_n-1}[X^{\tau_{j,n}}(Y^{\tau_{j+1,n}} - Y^{\tau_{j,n}}) - Y^{\tau_{j,n}}(X^{\tau_{j+1,n}} - X^{\tau_{j,n}})].$$

By using theorem 2.2.2(5) we get

$$V^{(n)}(X,Y) \to XY - ((X_-).Y) - ((Y_-).X) = [X,Y], \quad \textit{in ucp}.$$

that is

$$V^{(n)}(X,Y)_t \xrightarrow{P} X_tY_t - ((X_-).Y)_t - ((Y_-).X)_t = X_tY_t - \int_0^t X_{v-}dY_v - \int_0^t Y_{v-}dX_v = [X,Y]_t,$$

uniformly in t. $[X,Y] = ([X,Y]_t)_{t\geq 0}$ is called **quadratic covariation process** of X and Y. Since the quadratic variation process has (by definition) *cadlag* paths, we have the following

Proposition 2.3.3. $\Delta[X,Y]_t = \Delta X_t \Delta Y_t$.

***Proof*.** Since $[X,Y]_t = X_tY_t - ((X_-).Y)_t - ((Y_-).X)_t$ we have

$$\Delta[X,Y]_t = \Delta(X_tY_t - ((X_-).Y)_t - ((Y_-).X)_t) = \Delta(XY)_t - \Delta((X_-).Y)_t - \Delta((Y_-).X)_t$$
$$= \Delta(XY)_t - (X_{t-})\Delta Y_t - (Y_{t-})\Delta X_t,$$

by properties of stochastic integral. Moreover

$$\Delta X_t \Delta Y_t = (X_t - X_{t-})(Y_t - Y_{t-}) = X_tY_t - X_tY_{t-} - X_{t-}Y_t + X_{t-}Y_{t-} =$$
$$X_tY_t - X_{t-}Y_{t-} - (X_{t-})(Y_t - (Y_{t-}) - Y_{t-}(X_t - X_{t-}) = \Delta(XY)_t - (X_{t-})\Delta Y_t - Y_{t-})\Delta X_t,$$

so that $\Delta[X,Y]_t = \Delta X_t \Delta Y_t$, as required.

•

The quadratic covariation process of a semimartingale can be decomposed into a continuous part and a purely discontinuous part, i.e.

$$[X,Y]_t = [X,Y]_t^{(c)} + [X,Y]_t^{(d)},$$

where $[X,Y]_t^{(d)} = \sum_{s\le t}(\Delta X_s)(\Delta Y_s)$.

Now, suppose that the time horizon is fixed at $T > 0$, and let $\{0 = t_{0,n} < t_{1,n} < \cdots < t_{j_n,n} = T$ a sequence of non-random partitions of $[0,T]$, such that $\sup_{0\le j\le j_n-1}|t_{j+1,n} - t_{j,n}| \to 0$, as $n \to \infty$. Define **realized covariance** of X and Y on $[0,T]$ the quantity

$$v^{(n)}(X,Y)_T := \sum_{j=0}^{j_n-1}(X_{t_{j+1,n}} - X_{t_{j,n}})(Y_{t_{j+1,n}} - Y_{t_{j,n}}),$$

while the **realized covariance process** is given by, for each n, $v^{(n)}(X,Y) = (v^{(n)}(X,Y)_t)_{t\in[0,T]}$, where $v^{(n)}(X,Y)_t = \sum_{j=0}^{j_n-1}(X_{t_{j+1,n}\wedge t} - X_{t_{j,n}\wedge t})(Y_{t_{j+1,n}\wedge t} - Y_{t_{j,n}\wedge t})$. Theorem 2.2.2(5) tell us that

$$v^{(n)}(X,Y) \to [X,Y], \quad \textit{in ucp},$$

or

$$\sup_{s\le t}|v^{(n)}(X,Y)_s - [X,Y]_s| \xrightarrow{P} 0, \quad \textit{uniformly in } t.$$

2.4 Poisson processes, Lévy processes and Lévy measures

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\ge0}, P)$ be a filtered probability space.

Definition 2.4.1. *Let $N = (N_t)_{t\ge0}$ be a counting process defined by*

$$N_t = \sum_n 1_{\{\tau_n \le t\}}, \quad t \ge 0,$$

*with values in $\mathbb{N}$. We say that N is a **Poisson process** if*

- *the sequence $\{\tau_n, n \in \mathbb{N}\}$ is a sequence of stopping times such that $\tau_n < \tau_{n+1}$, $P-a.s.$, and $\sup_n \tau_n = \infty$, $P-a.s.$;*
- *(Independent increments) for any $s < t$, $N_t - N_s$ is independent of $\mathcal{F}_s$;*
- *(Stationary increments) for any $s < t \le u < v$ such that $t - s = v - u$ the distribution of $N_t - N_s$ is the same as that of $N_v - N_u$.*

The properties of stationarity and independence of increments characterize the Poisson process, in the sense that each counting process with independent and stationary increments is a Poisson process.

Remark 2.4.2. *The process $N = (N_t)_{t\geq 0}$ is adapted to the filtration $\mathcal{F}_t$ because*

$$\{N_t = n\} = \{\omega \in \Omega : \tau_n(\omega) \leq t < \tau_{n+1}(\omega)\} \in \mathcal{F}_t, \quad n = 0, 1, 2, ...$$

Moreover, since $\sup_n \tau_n = \infty$, $P - a.s.$, N is finite $P - a.s.$, that is, $P(N_t < \infty) = 1$, for every t.

Proposition 2.4.3. *Let N be a Poisson process, then for fixed $t \geq 0$*

$$P(N_t = n) = e^{-\lambda t}\frac{(\lambda t)^n}{n!}, \quad n = 0, 1, 2, ...,$$

for some $\lambda \geq 0$; in other words $N_t \sim Po(\lambda t)$ for every $t \geq 0$.

***Proof*.** We give the proof by steps as in Protter (1990). We first prove that $P(N_t = 0) = e^{-\lambda t}$, $\lambda \geq 0$. We have for $s \leq t$

$$\{\omega : N_t(\omega) = 0\} = \{\omega : N_s(\omega) = 0\} \cap \{\omega : N_t(\omega) - N_s(\omega) = 0\},$$

hence

$$P(N_t = 0) = P(\{N_s = 0\} \cap \{N_t - N_s = 0\}) = P(N_s = 0)P(N_t - N_s = 0) = P(N_s = 0)P(N_{t-s} = 0)$$

by independence and stationary increments. Now, if we set $\gamma(s) = P(N_s = 0)$ we obtain the equality $\gamma(t) = \gamma(s)\gamma(t-s)$: a possible solution is of exponential type, $\gamma(t) = e^{-\lambda t}$, for some $\lambda \geq 0$, unless we take $\gamma(t) = 0$ for all t. However, from $\gamma(t) = 0$ follows that $N_t = \infty$, $P - a.s.$, for all $t \geq 0$ which is a contradiction since N_t is finite, $P - a.s.$, therefore the unique solution is $P(N_t = 0) = e^{-\lambda t}$, as required.

Now, the second fact we need to prove is $P(N_t \geq 2) = o(t)$. Let $\delta(t) = P(N_t \geq 2)$. We have to show that $\delta(\frac{1}{n}) = o(\frac{1}{n})$ as $n \to \infty$. If we divide the interval $[0, 1]$ into n equally spaced subintervals, the probability that each subinterval contains at least two events is $\delta(\frac{1}{n})$. By independence and stationary increments, the number of subintervals which contain at least two events, S_n, is a binomial r.v. with parameters n and $\delta(\frac{1}{n})$. But, for each ω for n sufficiently large no subinterval has more one events by properties of the Poisson process. That is, $P(S_n(\omega) \to 0) = 1$, as $n \to \infty$. Hence, $\lim_n n\delta(\frac{1}{n}) = \lim_n E(S_n) = 0$, so that $\delta(\frac{1}{n}) = o(\frac{1}{n})$.

The third fact to show is that $\lim_{t\to 0}\frac{P(N_t=1)}{t} = \lambda$. We have

$$\lim_{t\to 0}\frac{P(N_t = 1)}{t} = \lim_{t\to 0}\frac{1 - P(N_t = 0) - P(N_t \geq 2)}{t} = \lim_{t\to 0}\frac{1 - e^{-\lambda t} - o(t)}{t} =$$

$$\lim_{t\to 0}\frac{1 - e^{-\lambda t}}{t} - \lim_{t\to 0}\frac{o(t)}{t} = \lambda,$$

which implies that $P(N_t = 1) = \lambda t e^{-\lambda t}$.

Finally, we consider the probability generating function $\psi(t) = E(z^{N_t})$. We can write

$$\psi(t+s) = E(z^{N_{t+s}}) = E(z^{N_{t+s}-N_s+N_s}) = E(z^{(N_{t+s}-N_s)+N_s}) =$$

$$E(z^{N_s})E(z^{N_{t+s}-N_s}) = E(z^{N_s})E(z^{N_t}) = \psi(s)\psi(t),$$

so that $\psi(t)$ is of the kind $\psi(t) = e^{t\phi(z)}$. To determine the explicit form of $\phi(z)$, we remark that $\frac{d}{dt}\psi(t)\mid_{t=0} = \phi(z)$; therefore

$$\phi(z) = \lim_{t\to 0}\frac{\psi(t) - 1}{t} = \lim_{t\to 0}\frac{E(z^{N_t}) - 1}{t} =$$

$$\lim_{t\to 0}\frac{P(N_t=0)+zP(N_t=1)+\sum_{k=2}^{\infty}z^kP(N_t=k)-1}{t}=$$

$$\lim_{t\to 0}\frac{P(N_t=0)-1}{t}+\lim_{t\to 0}\frac{zP(N_t=1)}{t}+\lim_{t\to 0}\frac{o(t)}{t}=$$

$$\lim_{t\to 0}\frac{e^{-\lambda t}-1}{t}+\lambda z+0=-\lambda+\lambda z,$$

hence $\psi(t)=e^{-\lambda t+\lambda z t}$. Then, by Taylor series expansion

$$\psi(t)=\sum_{k=0}^{\infty}z^kP(N_t=k)=e^{-\lambda t}e^{\lambda t z}=e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(\lambda t)^k z^k}{k!}$$

so that $P(N_t=k)=e^{-\lambda t}\frac{(\lambda t)^k}{k!}$, as required.

•

An immediate consequence of the above proposition is that $EN_t = Var(N_t) = \lambda t$. Then, by properties of increments, it easily follows that $N_t-\lambda t$ and $(N_t-\lambda t)^2-\lambda t$ are martingales w.r.t. the filtration $(\mathcal{F}_t)_{t\geq 0}$. In fact, for $s\leq t$

$$E[(N_t-\lambda t)-(N_s-\lambda s)|\mathcal{F}_s]=E[N_t-N_s|\mathcal{F}_s]-\lambda(t-s)=$$

$$E(N_t-N_s)-\lambda(t-s)=\lambda(t-s)-\lambda(t-s)=0,\quad P-a.s.$$

The most important properties of a Poisson process are the following.

1. The sample paths, $N_.(\omega):t\mapsto N_t(\omega)$, are *cadlag* and in particular are piecewise constants and increase by jumps of size 1;

2. $P(N_t=N_{t-})=1$, $\forall t\geq 0$;

3. $N=(N_t)_{t\geq 0}$ is continuous in probability, i.e.

$$\lim_{s\to t}P(|N_s-N_t|>\varepsilon)=0,\quad \forall t\geq 0,\ \forall\varepsilon>0.$$

Each counting process generates a counting measure on the space where it takes its values. In particular, they are generally known as random measures. For example, for each ω, the Poisson process yields a measure M on $\mathbb{R}_+$ defined as

$$M(\omega,A)=\#\{n\in\mathbb{N}:\tau_n(\omega)\in A\},\quad A\subset\mathbb{R}_+.$$

We immediately see that $N_t(\omega)=M(\omega,[0,t]):=\int_0^t M(\omega,ds)$.

Definition 2.4.4. *A mapping $\mu:\Omega\times\mathbb{E}\to\mathbb{N}$, where $\mathbb{E}\subset\mathbb{R}^d$ is a* ***Poisson random measure*** *if*

- *for almost all $\omega\in\Omega$, the function $\mu(\omega,\cdot):=\mu(A):A\mapsto\mu(\omega,A)$ is a Radon measure on $(\mathbb{E},\mathcal{B}_{\mathbb{E}})$, that is, $\mu(A)<\infty$ for any compact subset of $\mathbb{E}$;*

- *for every $A\in\mathcal{B}_{\mathbb{E}}$, the mapping $\mu(\cdot,A):=\mu_A(\omega):\omega\mapsto\mu(\omega,A)$ is a Poisson r.v. with parameter $\lambda(A)$ where λ is a Radon measure on $(\mathbb{E},\mathcal{B}_{\mathbb{E}})$, i.e.,*

$$P(\mu_A=n)=e^{-\lambda(A)}\frac{(\lambda(A))^n}{n!},\quad n=0,1,2,....$$

- *for any disjoint finite family of measurable sets $A_1,....,A_n$ the r.v.s $\mu_{A_1},....,\mu_{A_n}$ are independent.*

Let $\mathbb{R}_0 = \mathbb{R} - \{0\}$. We are interested in the Poisson r.m., μ, defined on $\Omega \times [0,T] \times \mathbb{R}_0$, for some $T \geq 0$. Let λ be the Lebesgue measure on $[0,T] \times \mathbb{R}_0$. By construction (see Cont-Tankov, 2004), μ can be described as the counting measure associated to a random sequence of points $\{(\tau_n, Y_n), n \in \mathbb{N}\}$ such that $(\tau_n, Y_n) \in [0,T] \times \mathbb{R}_0$ defined by

$$\mu = \sum_n 1_{[0,T]\times\mathbb{R}_0}(\tau_n, Y_n),$$

where $\{\tau_n, n \in \mathbb{N}\}$ is a sequence of stopping times and $\{Y_n, n \in \mathbb{N}\}$ is a sequence of $\mathcal{F}_{\tau_n}$-measurable r.v.s. Therefore, we have

$$\mu(\omega, [0,t] \times A) = \sum_n 1_{[0,t]\times A}(\tau_n(\omega), Y_n(\omega)) =$$

$$\#\{(\tau_n(\omega), Y_n(\omega)) \in [0,T] \times \mathbb{R}_0 : \tau_n(\omega) \in [0,t] \textit{ and } Y_n(\omega) \in A\},$$

for every $t \leq T$ and $A \in \mathcal{B}_{\mathbb{R}_0}$ by definition, the r.v. $\mu(\cdot, [0,t] \times A) = \mu_{[0,t]\times A}$ has a Poisson distribution with parameter $E[\mu_{[0,t]\times A}] = \lambda([0,t] \times A)$. Moreover, since for each ω, $\mu(\omega, \cdot)$ is a measure on $[0,T] \times \mathbb{R}_0$ we may define an integral w.r.t. it, for measurable functions g defined on $[0,T] \times \mathbb{R}_0$, $g : [0,T] \times \mathbb{R}_0 \to \mathbb{R}$. In particular, if g is simple, that is, of the form $g(t,x) = \sum_{i=0}^{m-1} \sum_{j=1}^{n} a_{ij} 1_{]\tau_i, \tau_{i+1}]} 1_{A_j}$, where $A_1, ..., A_n$ are disjoint sets of $\mathbb{R}_0$, such that $\lambda([0,T] \times A_j) < \infty$ $j = 1, 2, ..., n$, we define the integral of g w.r.t. μ as $(g.\mu) = \sum_{i=0}^{m-1} \sum_{j=1}^{n} a_{ij} \mu(]\tau_i, \tau_{i+1}] \times A_j)$. For positive measurable functions the definition is obtained by the monotone convergence theorem, $(g^{(n)}.\mu) \to (g.\mu)$, where $\{g^{(n)}, n \in \mathbb{N}\}$ is a nondecreasing sequence of simple functions approaching to g. When g is completely general we can define the integral provided that $(g.\lambda) = \int_{[0,T]\times\mathbb{R}_0} |g(t,x)| \lambda(dt, dx) < \infty$. Remark that $(g.\mu)$ is a r.v. with expectation $E(g.\mu) = (g.\lambda)$.

Proposition 2.4.5. *Let μ be a Poisson random measure defined on $[0,T] \times \mathbb{R}_0$ and let $g : [0,T] \times \mathbb{R}_0 \to \mathbb{R}$ be a measurable mapping satisfying $(g.\lambda) < \infty$; then the stochastic process*

$$X_t := (g.\tilde{\mu})_t = \int_0^t \int_{\mathbb{R}_0} g(s,x) \tilde{\mu}(ds, dx) =$$

$$\int_0^t \int_{\mathbb{R}_0} g(s,x) \mu(ds, dx) - \int_0^t \int_{\mathbb{R}_0} g(s,x) \lambda(ds, dx), \quad t \in [0,T],$$

is a martingale.

•

The random measure $\tilde{\mu} = \mu - \lambda$ which appears in proposition 2.4.5 is called **compensated Poisson random measure**: it makes $(g.\tilde{\mu})_t$ be a martingale.

Now, let $X = (X_t)_{t\in[0,T]}$ be a stochastic process with *cadlag* paths, and consider the sequence $\{(\tau_n, \Delta X_{\tau_n}), n \in \mathbb{N}\}$ where τ_n are the random jump times of X and $\Delta X_{\tau_n} = X_{\tau_n} - X_{\tau_{n-}}$ are the size of jumps. Remark that, by properties of *cadlag* functions $\Delta X_{\tau_n} \in \mathbb{R}_0$, and so we can write $\{(\tau_n, \Delta X_{\tau_n}), n \in \mathbb{N}\} \subset [0,T] \times \mathbb{R}_0$. Then

$$\chi_X([0,t] \times A) = \sum_n 1_{[0,t]\times A}(\tau_n, \Delta X_{\tau_n}) = \sum_{s\in[0,t]} 1_A(\Delta X_s), \quad t \in [0,T],$$

is a random measure and since it is derived by the jumps of the process X it will be called the **jump measure** associated to X. As above, we can define an integral w.r.t. χ_X; in particular, for measurable functions g we have

$$\int_0^T \int_{\mathbb{R}_0} g(t,x) \chi_X(dt, dx) = \sum_{t\in[0,T]} g(\Delta X_t).$$

If N is a Poisson process (which is *cadlag*), since the jumps have all size 1, the associated jump measure χ_N is defined by

$$\chi_N([0,t]\times A)=\begin{cases}\sum_{s\in[0,t]}1_A(1), & 1\in A;\\ 0, & 1\notin A.\end{cases}$$

Definition 2.4.6. *An adapted stochastic process $L=(L_t,\mathcal{F}_t)_{t\geq 0}$ with values in $\mathbb{R}$ is a **Lévy process** if*

1. *(Independent increments)for any $s<t$, L_t-L_s is independent of $\mathcal{F}_s$;*
2. *(Stationary increments) for any $s<t\leq u<v$ such that $t-s=v-u$ the distribution of L_t-L_s is the same as that of L_v-L_u;*
3. *$L_0=0$, $P-a.s.$;*
4. *it is continuous in probability, i.e.*

$$\lim_{s\to t}P(|L_s-L_t|>\varepsilon)=0,\quad \forall t\geq 0,\ \forall\varepsilon>0;$$

5. *the sample paths, $t\mapsto L_t(\omega)$, are* cadlag*, for every $t\geq 0$.*

It's not hard to prove the following

Proposition 2.4.7. *Let $L=(L_t,\mathcal{F}_t)_{t\geq 0}$ be a Lévy process. Then L_t has an infinitely divisible distribution $\forall t\geq 0$; conversely, if μ is an infinitely divisible distribution then there exists a Lévy process L such that the distribution of L_1 is μ.*

•

Since a Lévy process has *cadlag* paths the jumps $\Delta L_t=L_t-L_{t-}$ are defined and $\Delta L_t\in\mathbb{R}_0$. Suppose that $\sup_{t\geq 0}|\Delta L_t|<\infty$, $P-a.s.$. Let $A\in\mathcal{B}_{\mathbb{R}_0}$. Define

$$\tau_A^{(1)}=\inf\{t>0:\Delta L_t\in A\}$$

$$\tau_A^{(2)}=\inf\{t>\tau_A^{(1)}:\Delta L_t\in A\}$$

.....

$$\tau_A^{(n+1)}=\inf\{t>\tau_A^{(n)}:\Delta L_t\in A\}$$

It can be shown that $\{\tau_A^{(n)},n\in\mathbb{N}\}$ is a sequence of stopping times s.t. $\tau_A^{(1)}>0$, $P-a.s.$, and $\tau_A^{(n)}\to\infty$, $P-a.s.$. Define the counting measure

$$\kappa([0,t]\times A)=\sum_n 1_{\{\tau_A^{(n)}\leq t\}}=\sum_{s\leq t}1_A(\Delta L_s),\quad t\geq 0.$$

It is finite $P-a.s.$, and moreover, it has independent and stationary increments. Hence $\kappa([0,t]\times A)$ is a Poisson process and for a fixed t it is a Poisson random measure. Thus,

$$\int_0^t\int_A\kappa(dt,dx)=\sum_{s\leq t}1_A(\Delta L_s),$$

and

$$\int_0^t\int_A g(x)\kappa(dt,dx)=\sum_{s\leq t}g(\Delta L_s)1_A(\Delta L_s).$$

Definition 2.4.8. *The measure ν on $\mathbb{R}_0$ defined by*

$$\nu(A) = E[\kappa([0,1] \times A)] = E \sum_{s \in [0,1]} 1_A(\Delta L_s)$$

is called ***Lévy measure*** *of the Lévy process L.*

Remark 2.4.9. *Lévy measure ν is σ-finite; intuitively, it is the expectation of the number of jumps of L on $[0,1]$ whose size is in A.*

Moreover $E[\kappa([0,1] \times A)] = \nu(A)$ and by properties of the Poisson process we also have $E[\kappa([0,t] \times A)] = t\nu(A)$.

Proposition 2.4.10. *Let $g : [0,T] \times \mathbb{R}_0 \to \mathbb{R}$. We have*

$$E \int_0^T \int_{\mathbb{R}_0} g(t,x)\kappa(dt,dx) = \int_0^T \int_{\mathbb{R}_0} g(t,x)\nu(dx)dt.$$

Proof. Suppose that g is simple, that is, of the form $g(t,x) = \sum_{i=0}^{m-1} \sum_{j=1}^{n} a_{ij} 1_{]t_i,t_{i+1}] \times A_j}(t,x)$. Then

$$E \int_0^T \int_{\mathbb{R}_0} g(t,x)\kappa(dt,dx) = E \sum_{i=0}^{m-1} \sum_{j=1}^{n} a_{ij}\kappa(]t_i,t_{i+1}] \times A_j) =$$

$$\sum_{i=0}^{m-1} \sum_{j=1}^{n} a_{ij} E(\kappa(]t_i,t_{i+1}] \times A_j)) = \sum_{i=0}^{m-1} \sum_{j=1}^{n} a_{ij}(t_{i+1} - t_i)\nu(A_j) = \int_0^T \int_{\mathbb{R}_0} g(t,x)\nu(dx)dt.$$

For general functions $g(\cdot,\cdot)$, we proceed by approximation through a sequence of simple functions $\{g^{(n)}(\cdot,\cdot), n \in \mathbb{N}\}$ convergent to $g(\cdot,\cdot)$, as usual.

If $g(t,x) = f(x) = \sum_{j=1}^{n} a_j 1_{A_j}(x)$, therefore

$$E \int_0^T \int_{\mathbb{R}_0} f(x)\kappa(dt,dx) = \sum_{i=0}^{m-1} (t_{i+1} - t_i) \sum_{j=1}^{n} a_j \nu(A_j) = T \int_{\mathbb{R}_0} f(x)\nu(dx).$$

•

As an immediate consequence, we have

Proposition 2.4.11. *Let $f : \mathbb{R}_0 \to \mathbb{R}$. We have*

$$E(\int_0^T \int_{\mathbb{R}_0} f(x)\kappa(dt,dx) - T \int_{\mathbb{R}_0} f(x)\nu(dx))^2 = T \int_{\mathbb{R}_0} f^2(x)\nu(dx).$$

provided the last integral exists.

•

Proposition 2.4.12. *Let $A_1, A_2 \in \mathcal{B}_{\mathbb{R}_0}$ such that $A_1 \cap A_2 = \varnothing$. Then, the two processes*

$$J_t^{(1)} = \int_0^t \int_{A_1} x\kappa(dt,dx) = \sum_{s \le t} \Delta L_s 1_{A_1}(\Delta L_s)$$

and

$$J_t^{(2)} = \int_0^t \int_{A_2} x\kappa(dt,dx) = \sum_{s \le t} \Delta L_s 1_{A_2}(\Delta L_s),$$

are independent Lévy processes.

•

Theorem 2.4.13. *(Protter, 1990) Let $L = (L_t)_{t\geq 0}$ be a Lévy process with jumps bounded by $0 < M < \infty$, $\sup_t |\Delta L_t| < M$, $P-a.s.$. Let $Z_t = L_t - E(L_t)$. Then $Z = (Z_t)_{t\geq 0}$ is a martingale and $Z_t = Z_t^{(c)} + Z_t^{(d)}$ where $Z_t^{(c)}$ is a continuous martingale and*

$$Z_t^{(d)} = \int_0^t \int_{|x|\leq M} x(\kappa(dt,dx) - \nu(dx)dt)$$

is also a martingale. Moreover, $Z_t^{(c)}$ and $Z_t^{(d)}$ are independent Lévy processes.

•

Theorem 2.4.14. *(Jacod-Shiryaev, 1987) Let $L = (L_t)_{t\geq 0}$ be a Lévy process. Then, L has a decomposition*

$$L_t = W_t + \gamma t + \int_0^t \int_{|x|<1} x(\kappa(dt,dx) - \nu(dx)dt) + \sum_{0<s\leq t} \Delta L_s 1_{\{|\Delta L_s|\geq 1\}},$$

where $W = (W_t)_{t\geq 0}$ is a Wiener process, $\kappa([0,t]\times A) = \int_0^t \int_A \kappa(dt,dx)$ is a Poisson process independent of W for any $A \in \mathcal{B}_{\mathbb{R}_0}$, $\kappa([0,t]\times A) \perp \kappa([0,t]\times B)$ if A and B are disjoint, and the measure ν such that $E[\kappa([0,1]\times A)] = \nu(A)$ has the property $\int_{\mathbb{R}_0} (1\wedge x^2)\nu(dx) < \infty$.

•

Theorem 2.4.15. *(Lévy-Khintchine formula) Let L be a Lévy process with Lévy measure ν. Then, the Fourier transform of L is given by*

$$\varphi_t(z) = E(e^{izL_t}) = e^{-t\Lambda(z)}$$

where

$$\Lambda(z) = \frac{\sigma^2}{2} z^2 - i\gamma z + \int_{|x|\geq 1} (1 - e^{izx})\nu(dx) + \int_{|x|<1} (1 - e^{izx} + izx)\nu(dx).$$

•

Since ν is not necessarily a finite measure, that is, it is possible to have $\nu(\mathbb{R}_0) = \infty$, L can have an infinite number of small jumps on a compact interval $[0,T]$, and so the sum of jumps is an infinite series whose convergence requires conditions on ν, in particular

$$\int_{|x|\leq 1} x^2 \nu(dx) < \infty, \qquad \int_{|x|\geq 1} \nu(dx) < \infty.$$

Then, we have the celebrate **Lévy-Ito decomposition**

$$L_t = W_t + \gamma t + \int_0^t \int_{|x|\geq 1} x\kappa(dt,dx) + \int_0^t \int_{\varepsilon\leq |x|<1} x(\kappa(dt,dx) - \nu(dx)dt),$$

where the terms are independent and the convergence of the last integral is almost sure and uniform in $t \in [0,T]$. The Lévy-Ito decomposition establishes that *for every Lévy process there exist $\gamma \in \mathbb{R}$, $\sigma > 0$ and a positive measure ν that uniquely determine its distribution.* The triplet (σ, γ, ν) is called **Lévy triplet** or **characteristic triplet** of the process L.

Definition 2.4.16. *We define* ***compound Poisson process*** *with intensity β and jump size distribution G the stochastic process*

$$Y_t = \sum_{j=1}^{N_t} Z_j,$$

where $Z_j \sim G$, $j = 1, 2, ..., N_t$ and N_t is a Poisson process with parameter β.

The following proposition gives a characterization of the compound Poisson processes in terms of Lévy processes: they are the only Lévy processes with piecewise constant trajectories.

Proposition 2.4.17. *The process $Y = (Y_t)_{t\geq 0}$ is a compound Poisson if and only if it is a Lévy process with piecewise constant sample paths. Moreover, its jump measure is a Poisson random measure on $\mathbb{R}_+ \times \mathbb{R}$ with intensity $\nu(dx)dt = \beta G(dx)dt$, in other words, Y can be written as*

$$Y_t = \int_0^t \int_{\mathbb{R}} x\chi_Y(dt, dx) = \sum_{s\leq t} \Delta Y_s,$$

where, χ_Y is a Poisson random measure with intensity $\beta G(dx)dt$.

•

Since the condition $\int_{|x|\geq 1} \nu(dx) < \infty$ in the Lévy-Ito decomposition implies that the number of jumps with absolute value larger than 1 is finite, we can write the first integral as a compound Poisson process

$$\int_0^t \int_{|x|\geq 1} x\kappa(dt, dx) = \sum_{s\leq t:|\Delta L_s|\geq 1} \Delta L_s.$$

Now, suppose we have a bivariate Lévy process $(L_t^{(1)}, L_t^{(2)})$. In this work we are interested in its dependence structure. Here, we present two results relative to the marginal distributions and to the independence of the components $L_t^{(1)}$ and $L_t^{(2)}$ which emphasize the role of the Lévy measure.

Proposition 2.4.18. *Let $(L_t^{(1)}, L_t^{(2)})$ be a bivariate Lévy process with characteristic triplet $(\boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\nu})$. (Here, $\boldsymbol{\Sigma}$ is a positive definite matrix, $\boldsymbol{\Gamma}$ is a vector of $\mathbb{R}^2$ and $\boldsymbol{\nu}$ is positive measure on $\mathbb{R}^2_0$). Then $L_t^{(1)}$ has characteristic triplet $(\Sigma^{(1)}, \Gamma^{(1)}, \nu^{(1)})$ where*

$$\Sigma^{(1)} = \Sigma_{11},$$

$$\Gamma^{(1)} = \Gamma_1 + \int_{\mathbb{R}^2} x(1_{x^2\leq 1} - 1_{x^2+y^2\leq 1})\boldsymbol{\nu}(dx, dy),$$

$$\nu^{(1)}(A) = \boldsymbol{\nu}(A \times \mathbb{R}), \quad \forall A \in \mathcal{B}_{\mathbb{R}}.$$

•

Proposition 2.4.19. *Let $(L_t^{(1)}, L_t^{(2)})$ be a bivariate Lévy process with characteristic triplet $(\boldsymbol{\Sigma}, \boldsymbol{\Gamma}, \boldsymbol{\nu})$. Then $L_t^{(1)}$ and $L_t^{(1)}$ are independent if and only if the support of $\boldsymbol{\nu}$ belongs to the set $\{(x, y) \in \mathbb{R}^2 : xy = 0\}$, that is, if and only if never jump together. Moreover*

$$\boldsymbol{\nu}(A) = \nu^{(1)}(A^{(1)}) + \nu^{(2)}(A^{(2)}),$$

where $A^{(1)} = \{x : (x, 0) \in A\}$ and $A^{(2)} = \{y : (0, y) \in A\}$, and $\nu^{(1)}$ and $\nu^{(2)}$ are the Lévy measures of $L_t^{(1)}$ and $L_t^{(1)}$.

•

Notes

The most results presented in this chapter can be found in Karatzas-Shreve (1987), Krylov (2002), Protter (1990), Cont-Tankov (2004), Sato (1999).

Chapter 3

Preliminary results in continuous case

"...sorgon cosi' tue dive
membra dall'egro talamo,
e in te belta' rivive,
l'aurea beltade, ond'ebbero
ristoro unico a' mali
le nate a vaneggiar menti mortali. "
(U. Foscolo, All'amica risanata)

3.1 Introduction

This chapter provides some preliminary results due to Barndorff-Nielsen and Shephard (Barndorff-Nielsen and Shephard 2003, 2004c) necessary to derive our main theorems (chapter 4). They estimate the integrated covariation between two diffusion processes in absence of jumps by using the realized covariation and provide an asymptotic distributional analysis of it based on a fixed period of time (e.g. a day or week) allowing the number of high frequency returns during this period to go to infinity.

3.2 Some preliminary lemmas

Before showing the main propositions of Barndorff-Nielsen and Shephard, we need some useful lemmas which we prove here.

If $\{\pi_n, n \in \mathbb{N}\}$ is a sequence of equally spaced partitions, i.e., $\pi_n = \{0 = t_{0,n} < t_{1,n} < \cdots < t_{n,n} = T\}$ with $h_n = t_{j,n} - t_{j-1,n}$, such that $h_n \to 0$ as $n \to \infty$, we introduce the following process

$$v_{r,l}^{(n)}(X,Y)_T := h_n^{1-\frac{r+l}{2}} \sum_{j=1}^{n} (X_{t_{j,n}} - X_{t_{j-1,n}})^r (Y_{t_{j,n}} - Y_{t_{j-1,n}})^l = h_n^{1-\frac{r+l}{2}} \sum_{j=1}^{n} (\Delta_{j,n}X)^r (\Delta_{j,n}Y)^l,$$

for any couple of stochastic processes X and Y defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, P)$ and adapted to $(\mathcal{F})_{t\in[0,T]}$.

Lemma 3.2.1. *Let $W^{(1)}$ and $W^{(2)}$ be two Wiener processes such that $Cov(W_t^{(1)}, W_t^{(2)}) = \rho t$ then*

$$v_{1,1}^{(n)}(W^{(1)}, W^{(2)})_T \to \rho T, \quad P - a.s.$$

as $n \to \infty$.

Proof. Since $\Delta_j W^{(q)} \sim N(0,h)$, $q=1,2$, we have $\Delta_{j,n} W^{(q)} \stackrel{D}{=} \sqrt{h} Z_{j,n}^{(q)}$ where $Z_{j,n}^{(q)}$ is a standard normal r.v., for each j and each n and $Corr(Z_1^{(1)}, Z_1^{(2)}) = \rho$. Moreover, $(\Delta_{j,n} W^{(1)} \Delta_{j,n} W^{(2)})_{j=1,\dots,n}$ is a sequence of independent identically distributed random variables such that

$$E(\Delta_{j,n} W^{(1)} \Delta_{j,n} W^{(2)}) = \rho \sqrt{E(\Delta_{j,n} W^{(1)})^2 E(\Delta_{j,n} W^{(2)})^2} = \rho h$$

Then, by the Strong Law of Large Numbers (SLLN)

$$v_{1,1}^{(n)}(W^{(1)}, W^{(2)})_T = \sum_{j=1}^n \Delta_{j,n} W^{(1)} \Delta_{j,n} W^{(2)} = \sum_{j=1}^n \sqrt{h} Z_{j,n}^{(1)} \sqrt{h} Z_{j,n}^{(2)} = h \sum_{j=1}^n Z_{j,n}^{(1)} Z_{j,n}^{(2)} =$$

$$nh \frac{1}{n} \sum_{j=1}^n Z_{j,n}^{(1)} Z_{j,n}^{(2)} \to \rho T, \quad P-a.s.$$

•

Analogously, we have

Lemma 3.2.2. *Let $W^{(1)}$ and $W^{(2)}$ be two independent Wiener processes, then*

$$v_{r,l}^{(n)}(W^{(1)}, W^{(2)})_T \to T \mu_{1,r} \mu_{2,l}, \quad P-a.s.$$

as $n \to \infty$, where $\mu_{1,r}$ and $\mu_{2,l}$ are the r-th and l-th moments of two standard Gaussian r.v.s.

Proof. As above, by the SLLN

$$v_{r,l}^{(n)}(W^{(1)}, W^{(2)})_T = h_n^{1-\frac{r+l}{2}} \sum_{j=1}^n (\Delta_{j,n} W^{(1)})^r (\Delta_{j,n} W^{(2)})^l = h_n^{1-\frac{r+l}{2}} h^{\frac{r}{2}} h^{\frac{l}{2}} \sum_{j=1}^n (Z_{j,n}^{(1)})^r (Z_{j,n}^{(2)})^l$$

$$= nh_n \frac{1}{n} \sum_{j=1}^n (Z_{j,n}^{(1)})^r (Z_{j,n}^{(2)})^l \to TE[(Z_{1,1}^{(1)})^r (Z_{1,1}^{(2)})^l] = TE(Z_{1,1}^{(1)})^r E(Z_{1,1}^{(2)})^l = T\mu_{1,r}\mu_{2,l}, \quad P-a.s.$$

since $\{Z_{j,n}^{(1)},\ j=1,\dots,n,\ n=1,2,\dots\} \sim iidN(0,1)$ and $\{Z_{j,n}^{(2)},\ j=1,\dots,n,\ n=1,2,\dots\} \sim iidN(0,1)$ and moreover $Z_{j,n}^{(1)}$ is independent of $Z_{j,n}^{(2)}$ for each $j=1,\dots,n$ and each n.

•

Remark 3.2.3. *If at least one between the exponents r and l is odd the previous limit is zero. If $W^{(1)} = W^{(2)}$ we get anyway that*

$$v_{r,l}^{(n)}(W^{(1)}, W^{(1)})_T \to T\mu_{r+l}, \quad P-a.s.$$

where, obviously, μ_{r+l} is the moment of order $r+l$ of a standard Gaussian r.v..

Lemma 3.2.4. *Let $\varphi = (\varphi_t)_{t\in[0,T]}$ be an adapted stochastic process pathwise Riemann integrable and such that pathwise $0 < \underline{\varphi}_T < \overline{\varphi}_T < \infty$, where $\underline{\varphi}_T = \inf_{t\in[0,T]} \varphi_t$ and $\overline{\varphi}_T = \sup_{t\in[0,T]} \varphi_t$; then, as $n \to \infty$*

$$h_n^{1-r} \sum_{j=1}^n (\Delta_{j,n}\varphi)^r \to \int_0^T \varphi_s^r ds, \quad pathwise,$$

for every $r > 0$.

Proof. Since there exists $\varrho_{j,n}$ such that $\inf_{t\in[t_{j-1,n},t_{j,n}]} \varphi_t \le \varrho_{j,n} \le \sup_{t\in[t_{j-1,n},t_{j,n}]} \varphi_t$ and

$$\Delta_{j,n}\varphi = \int_{t_{j-1,n}}^{t_{j,n}} \varphi_s ds = (t_{j,n} - t_{j-1,n})\varrho_{j,n} = h_n \varrho_{j,n},$$

then

$$h_n^{1-r} \sum_{j=1}^n (\Delta_{j,n}\varphi)^r = h_n^{1-r} \sum_{j=1}^n h_n^r \varrho_{j,n}^r = \sum_{j=1}^n h_n \varrho_{j,n}^r = \sum_{j=1}^n \varrho_{j,n}^r (t_{j,n} - t_{j-1,n}) \to \int_0^T \varphi_s^r ds,$$

by the (pathwise) Riemann integrability of φ.

•

Lemma 3.2.5. *Let $\varphi_1 = (\varphi_{1t})_{t\in[0,T]}$ and $\varphi_2 = (\varphi_{2t})_{t\in[0,T]}$ be two adapted stochastic processes which satisfy pathwise conditions of lemma 3.2.4; then, as $n \to \infty$*

$$h_n^{1-r-l}\sum_{j=1}^{n}(\Delta_{j,n}\varphi_1)^r(\Delta_{j,n}\varphi_2)^l \to \int_0^T \varphi_{1s}^r\varphi_{2s}^l ds, \quad \textit{pathwise},$$

for every $r, l > 0$.

Proof. As above if $\inf_{t\in[t_{j-1,n},t_{j,n}]}\varphi_{1t} \le \varrho_{j,n} \le \sup_{t\in[t_{j-1,n},t_{j,n}]}\varphi_{1t}$ and $\inf_{t\in[t_{j-1,n},t_{j,n}]}\varphi_{1t} \le \varsigma_{j,n} \le \sup_{t\in[t_{j-1,n},t_{j,n}]}\varphi_{2t}$, then

$$h_n^{1-r-l}\sum_{j=1}^{n}(\Delta_{j,n}\varphi_1)^r(\Delta_{j,n}\varphi_2)^l = h_n^{1-r-l}\sum_{j=1}^{n}h_n^r\varrho_{j,n}^r h_n^l\varsigma_{j,n}^l = \sum_{j=1}^{n}h_n\varrho_{j,n}^r\varsigma_{j,n}^l =$$

$$\sum_{j=1}^{n}\varrho_{j,n}^r\varsigma_{j,n}^l(t_{j,n}-t_{j-1,n}) \to \int_0^T \varphi_{1s}^r\varphi_{2s}^l ds.$$

•

The following lemma will be used in the proof of proposition 3.3.4.

Lemma 3.2.6. *Let $\varphi_1 = (\varphi_{1t})_{t\in[0,T]}$ and $\varphi_2 = (\varphi_{2t})_{t\in[0,T]}$ be* cadlag *adapted stochastic processes satisfying conditions of lemma 3.2.4; then*

$$h_n^{-1}[\sum_{j=1}^{n-1}(\Delta_{j,n}\varphi_1)^{1/2}(\Delta_{j,n}\varphi_2)^{1/2}(\Delta_{j+1,n}\varphi_1)^{1/2}(\Delta_{j+1,n}\varphi_2)^{1/2} - \sum_{j=1}^{n}(\Delta_{j,n}\varphi_1)(\Delta_{j,n}\varphi_2)] = O(h_n).$$

Proof. To simplify the notation we write t_j instead of $t_{j,n}$ and $\Delta_j\varphi$ instead of $\Delta_{j,n}\varphi$. We essentially follow Barndorff-Nielsen and Shephard (2004a). We can write

$$\sum_{j=1}^{n-1}(\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}(\Delta_{j+1}\varphi_1)^{1/2}(\Delta_{j+1}\varphi_2)^{1/2} - \sum_{j=1}^{n}(\Delta_j\varphi_1)(\Delta_j\varphi_2) =$$

$$\sum_{j=1}^{n-1}[(\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}(\Delta_{j+1}\varphi_1)^{1/2}(\Delta_{j+1}\varphi_2)^{1/2} - (\Delta_j\varphi_1)(\Delta_j\varphi_2)] - (\Delta_n\varphi_1)(\Delta_n\varphi_2) =$$

$$\sum_{j=1}^{n-1}(\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}[(\Delta_{j+1}\varphi_1)^{1/2}(\Delta_{j+1}\varphi_2)^{1/2} - (\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}] - (\Delta_n\varphi_1)(\Delta_n\varphi_2) =$$

$$\sum_{j=1}^{n-1}\frac{(\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}[(\Delta_{j+1}\varphi_1)(\Delta_{j+1}\varphi_2) - (\Delta_j\varphi_1)(\Delta_j\varphi_2)]}{(\Delta_{j+1}\varphi_1)^{1/2}(\Delta_{j+1}\varphi_2)^{1/2} + (\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}} - (\Delta_n\varphi_1)(\Delta_n\varphi_2) =$$

$$\frac{1}{2}\sum_{j=1}^{n-1}\left[\frac{2(\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}}{(\Delta_{j+1}\varphi_1)^{1/2}(\Delta_{j+1}\varphi_2)^{1/2} + (\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}} - 1\right][(\Delta_{j+1}\varphi_1)(\Delta_{j+1}\varphi_2) - (\Delta_j\varphi_1)(\Delta_j\varphi_2)] +$$

$$+\frac{1}{2}\sum_{j=1}^{n-1}[(\Delta_{j+1}\varphi_1)(\Delta_{j+1}\varphi_2) - (\Delta_j\varphi_1)(\Delta_j\varphi_2)] - (\Delta_n\varphi_1)(\Delta_n\varphi_2) =$$

$$-\frac{1}{2}\sum_{j=1}^{n-1}\frac{(\Delta_{j+1}\varphi_1)^{1/2}(\Delta_{j+1}\varphi_2)^{1/2} - (\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}}{(\Delta_{j+1}\varphi_1)^{1/2}(\Delta_{j+1}\varphi_2)^{1/2} + (\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}}(\Delta_{j+1}\varphi_1)(\Delta_{j+1}\varphi_2) - (\Delta_j\varphi_1)(\Delta_j\varphi_2) +$$

$$+\frac{1}{2}[(\Delta_n\varphi_1)(\Delta_n\varphi_2) - (\Delta_1\varphi_1)(\Delta_1\varphi_2)] - (\Delta_n\varphi_1)(\Delta_n\varphi_2) =$$

$$-\frac{1}{2}\sum_{j=1}^{n-1}\frac{[(\Delta_{j+1}\varphi_1)(\Delta_{j+1}\varphi_2)-(\Delta_j\varphi_1)(\Delta_j\varphi_2)]^2}{[(\Delta_{j+1}\varphi_1)^{1/2}(\Delta_{j+1}\varphi_2)^{1/2}+(\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}]^2}-\frac{1}{2}[(\Delta_n\varphi_1)(\Delta_n\varphi_2)+(\Delta_1\varphi_1)(\Delta_1\varphi_2)].$$

Now, let $\phi_{qj}^2 = h^{-1}\Delta_j\varphi_q$, $q=1,2$. ϕ_{qj}^2 is bounded and strictly positive because

$$\phi_{qj}^2 = h^{-1}\Delta_j\varphi_q \le h^{-1}h\overline{\varphi}_{qT} < \infty, \quad \phi_{qj}^2 = h^{-1}\Delta_j\varphi_q \ge h^{-1}h\underline{\varphi}_{qT} > 0.$$

So that

$$-\frac{1}{2}\sum_{j=1}^{n-1}\frac{[(\Delta_{j+1}\varphi_1)(\Delta_{j+1}\varphi_2)-(\Delta_j\varphi_1)(\Delta_j\varphi_2)]^2}{[(\Delta_{j+1}\varphi_1)^{1/2}(\Delta_{j+1}\varphi_2)^{1/2}+(\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}]^2}-\frac{1}{2}[(\Delta_n\varphi_1)(\Delta_n\varphi_2)+(\Delta_1\varphi_1)(\Delta_1\varphi_2)] =$$

$$-\frac{1}{2}\sum_{j=1}^{n-1}\frac{(h\phi_{1,j+1}^2h\phi_{2,j+1}^2-h\phi_{1j}^2h\phi_{2j}^2)^2}{(h^{1/2}\phi_{1,j+1}h^{1/2}\phi_{2,j+1}+h^{1/2}\phi_{1j}h^{1/2}\phi_{2j})^2}-\frac{1}{2}(h^2\phi_{1n}^2\phi_{2n}^2+h^2\phi_{11}^2\phi_{21}^2) =$$

$$-\frac{h^2}{2}\left[\sum_{j=1}^{n-1}\frac{(\phi_{1,j+1}^2\phi_{2,j+1}^2-\phi_{1j}^2\phi_{2j}^2)^2}{(\phi_{1,j+1}\phi_{2,j+1}+\phi_{1j}\phi_{2j})^2}+(\phi_{1n}^2\phi_{2n}^2-\phi_{11}^2\phi_{21}^2)\right],$$

hence

$$h^{-1}[\sum_{j=1}^{n-1}(\Delta_j\varphi_1)^{1/2}(\Delta_j\varphi_2)^{1/2}(\Delta_{j+1}\varphi_1)^{1/2}(\Delta_{j+1}\varphi_2)^{1/2}-\sum_{j=1}^{n}(\Delta_j\varphi_1)(\Delta_j\varphi_2)] =$$

$$-\frac{h}{2}\left[\sum_{j=1}^{n-1}\frac{(\phi_{1,j+1}^2\phi_{2,j+1}^2-\phi_{1j}^2\phi_{2j}^2)^2}{(\phi_{1,j+1}\phi_{2,j+1}+\phi_{1j}\phi_{2j})^2}+(\phi_{1n}^2\phi_{2n}^2+\phi_{11}^2\phi_{21}^2)\right].$$

Now, by assumption

$$0 < \inf_{0\le t\le T}\varphi_{qt} \le \sup_{0\le t\le T}\varphi_{qt} < \infty, \quad q=1,2$$

and recalling that $\Delta_j\varphi_q = \int_{t_{j-1}}^{t_j}\varphi_{qs}ds$ we can write

$$0 < \inf_{0\le t\le T}\varphi_{qt} \le \inf\phi_{qj}^2 \le \sup\phi_{qj}^2 \le \sup_{0\le t\le T}\varphi_{qt} < \infty, \quad q=1,2,$$

uniformly in j. Therefore the quantity $(\phi_{1n}^2\phi_{2n}^2+\phi_{11}^2\phi_{21}^2)$ is bounded from above. Furthermore

$$(\phi_{1,j+1}\phi_{2,j+1}+\phi_{1j}\phi_{2j})^2 = (\phi_{1,j+1}^2\phi_{2,j+1}^2+\phi_{1j}^2\phi_{2j}^2+2\phi_{1,j+1}\phi_{2,j+1}\phi_{1j}\phi_{2j}) \ge$$

$$2\phi_{1,j+1}\phi_{2,j+1}\phi_{1j}\phi_{2j} \ge 2\underline{\varphi}_{1T}^2\underline{\varphi}_{2T}^2 > 0$$

so that

$$\sum_{j=1}^{n-1}\frac{(\phi_{1,j+1}^2\phi_{2,j+1}^2-\phi_{1j}^2\phi_{2j}^2)^2}{(\phi_{1,j+1}\phi_{2,j+1}+\phi_{1j}\phi_{2j})^2} \le \frac{1}{2\underline{\varphi}_{1T}^2\underline{\varphi}_{2T}^2}\sum_{j=1}^{n-1}(\phi_{1,j+1}^2\phi_{2,j+1}^2-\phi_{1j}^2\phi_{2j}^2)^2 < \infty,$$

(see Barndorff-Nielsen and Shephard (2003)). The desired result follows.

•

Lemma 3.2.7. *Let $W^{(1)} = (W_t^{(1)})_{t\in[0,T]}$ and $W^{(2)} = (W_t^{(2)})_{t\in[0,T]}$ be independent Wiener processes and let $\varphi^{(1)} = (\varphi_t^{(1)})_{t\in[0,T]}$ and $\varphi^{(2)} = (\varphi_t^{(2)})_{t\in[0,T]}$ two* cadlag *and adapted processes both satisfying conditions of lemma 3.2.4. Moreover, let r, l positive integers. Then*

$$v_{r,l}^{(n)}((\varphi^{(1)}.W^{(1)}),(\varphi^{(2)}.W^{(2)}))_T \xrightarrow{P} \mu_{1,r}\mu_{2,l}\int_0^T(\varphi_t^{(1)})^r(\varphi_t^{(2)})^l dt$$

***Proof*.** We prove the statement for simple processes, that is, $\varphi^{(1)}$ and $\varphi^{(2)}$ are of the form

$$\varphi_t^{(1)}(\omega) := \xi_0^{(1)}(\omega)1_{\{0\}}(t) + \sum_{i=1}^{m} \xi_i^{(1)}(\omega)1_{]\frac{i-1}{m}T,\frac{i}{m}T]}(t), \quad t \in [0,T],\ \omega \in \Omega,$$

$$\varphi_t^{(2)}(\omega) := \xi_0^{(2)}(\omega)1_{\{0\}}(t) + \sum_{i=1}^{m} \xi_i^{(2)}(\omega)1_{]\frac{i-1}{m}T,\frac{i}{m}T]}(t), \quad t \in [0,T],\ \omega \in \Omega.$$

We can choose the same finite partition $\{\frac{iT}{m},\ i = 1, ..., m\}$ of $[0,T]$ for both processes without loss of generality. Hence

$$(\varphi^{(q)}.W^{(q)})_T = \sum_{i=1}^{m} \xi_i^{(1)}(W^{(q)}_{\frac{i}{m}T} - W^{(q)}_{\frac{i-1}{m}T}), \quad q = 1, 2,$$

and

$$\Delta_{j,n}(\varphi^{(q)}.W^{(q)}) = \xi_i^{(q)}(\Delta_{j,n}W^{(q)}), \quad if\ \frac{i-1}{m}T < t_{j-1,n} < t_{j,n} < \frac{i}{m}T, \quad i = 1, ..., m,\ j = 1, ..., n.$$

Now, assuming that each point $\frac{iT}{m}$, $i = 1, ..., m$, belongs to any partition $\pi_n = \{0 = t_{0,n} < t_{1,n} < \cdots < t_{j_n,n} = T\}$, we may write

$$v_{r,l}^{(n)}((\varphi^{(1)}.W^{(1)}), (\varphi^{(2)}.W^{(2)}))_T = h_n^{1-\frac{r+l}{2}} \sum_{j=1}^{n} [\Delta_{j,n}(\varphi^{(1)}.W^{(1)})]^r [(\Delta_{j,n}(\varphi^{(2)}.W^{(2)})]^l =$$

$$h_n^{1-\frac{r+l}{2}} \sum_{j=1}^{\frac{T}{m}} [\Delta_{j,n}(\varphi^{(1)}.W^{(1)})]^r [(\Delta_{j,n}(\varphi^{(2)}.W^{(2)})]^l + \cdots +$$

$$+ h_n^{1-\frac{r+l}{2}} \sum_{j=\frac{m-1}{m}T}^{T} [\Delta_{j,n}(\varphi^{(1)}.W^{(1)})]^r [(\Delta_{j,n}(\varphi^{(2)}.W^{(2)})]^l =$$

$$h_n^{1-\frac{r+l}{2}} \sum_{i=1}^{m} \sum_{j=\frac{i-1}{m}T}^{\frac{i}{m}T} [\Delta_{j,n}(\varphi^{(1)}.W^{(1)})]^r [(\Delta_{j,n}(\varphi^{(2)}.W^{(2)})]^l.$$

Then

$$v_{r,l}^{(n)}((\varphi^{(1)}.W^{(1)}), (\varphi^{(2)}.W^{(2)}))_T = h_n^{1-\frac{r+l}{2}} \sum_{i=1}^{m} \sum_{j=\frac{i-1}{m}T}^{\frac{i}{m}T} [\Delta_{j,n}(\varphi^{(1)}.W^{(1)})]^r [(\Delta_{j,n}(\varphi^{(2)}.W^{(2)})]^l =$$

$$h_n^{1-\frac{r+l}{2}} \sum_{i=1}^{m} \sum_{j=\frac{i-1}{m}T}^{\frac{i}{m}T} [(\xi_i^{(1)})^r(\Delta_{j,n}W^{(1)})^r][(\xi_i^{(2)})^l(\Delta_{j,n}W^{(2)})^l] =$$

$$\sum_{i=1}^{m} (\xi_i^{(1)})^r(\xi_i^{(2)})^l [h_n^{1-\frac{r+l}{2}} \sum_{j=\frac{i-1}{m}T}^{\frac{i}{m}T} (\Delta_{j,n}W^{(1)})^r(\Delta_j W^{(2)})^l] =$$

$$\sum_{i=1}^{m} (\xi_i^{(1)})^r(\xi_i^{(2)})^l v_{r,l}^{(n)}(W^{(1)}, W^{(2)})_{[\frac{i-1}{m}T,\frac{i}{m}T]} \xrightarrow{P}$$

$$\xrightarrow{P} \mu_{1,r}\mu_{2,l} \sum_{i=1}^{m} (\xi_i^{(1)})^r(\xi_i^{(2)})^l (\frac{i}{m}T - \frac{i-1}{m}T) = \mu_{1,r}\mu_{2,l} \int_0^T (\varphi_t^{(1)})^r(\varphi_t^{(2)})^l dt,$$

having used lemma 3.2.2. For general progressively measurable processes we can invoke approximating results presented in chapter 2.

•

The previous result holds even if the factors in the sum are more than two provided the Wiener integrators, giving the considered stochastic integrals are at most two. For example, we can apply proposition 3.2.7 to quantities of the type

$$h_n^{1-\frac{r+l+d}{2}} \sum_{j=1}^{n} [\Delta_{j,n}(\varphi^{(1)}.W^{(1)})]^r [(\Delta_{j,n}(\varphi^{(2)}.W^{(2)})]^l [\Delta_{j,n}(\varphi^{(3)}.W^{(1)})]^d,$$

where $\varphi^{(1)}$, $\varphi^{(2)}$ and $\varphi^{(3)}$ are progressively measurable processes. In fact, reasoning as in the proof of the proposition we get

$$h_n^{1-\frac{r+l+d}{2}} \sum_{j=1}^{n} [\Delta_{j,n}(\varphi^{(1)}.W^{(1)})]^r [(\Delta_{j,n}(\varphi^{(2)}.W^{(2)})]^l [\Delta_{j,n}(\varphi^{(3)}.W^{(1)})]^d =$$

$$\sum_{i=1}^{m} (\xi_i^{(1)})^r (\xi_i^{(2)})^l (\xi_i^{(3)})^d v_{r+d,l}^{(n)}(W^{(1)}, W^{(2)})_{[\frac{i-1}{m}T, \frac{i}{m}T]} \xrightarrow{P}$$

$$\xrightarrow{P} \mu_{1,r+d}\mu_{2,l} \sum_{i=1}^{m} (\xi_i^{(1)})^r (\xi_i^{(2)})^l (\xi_i^{(3)})^d (\frac{i}{m}T - \frac{i-1}{m}T) = \mu_{1,r+d}\mu_{2,l} \int_0^T (\varphi_t^{(1)})^r (\varphi_t^{(2)})^l (\varphi_t^{(3)})^d dt.$$

3.3 Main results in case of diffusion processes

Let $(\Omega, \mathcal{F}, (\mathcal{F})_{t\in[0,T]}, P)$ be a filtered probability space and let $X^{(1)} = (X_t^{(1)})_{t\in[0,T]}$ and $X^{(2)} = (X_t^{(2)})_{t\in[0,T]}$ be two martingales defined on it of the form

$$\begin{aligned} X_t^{(1)} &= \int_0^t \sigma_s^{(1)} dW_s^{(1)} = (\sigma^{(1)}.W^{(1)})_t, \quad t \in [0,T] \\ X_t^{(2)} &= \int_0^t \sigma_s^{(2)} dW_s^{(2)} = (\sigma^{(2)}.W^{(2)})_t, \quad t \in [0,T] \end{aligned}$$

where $W^{(1)} = (W_t^{(1)})_{t\in[0,T]}$ is a Wiener process and

$$W_t^{(2)} = \rho W_t^{(1)} + \sqrt{1-\rho^2} W_t^{(3)},$$

with $W^{(3)} = (W_t^{(3)})_{t\in[0,T]}$ independent of $W^{(1)}$, and $\rho \in [-1,1]$. More explicitly

$$X_t^{(1)} = (\sigma^{(1)}.W^{(1)})_t$$

$$X_t^{(2)} = (\rho\sigma^{(2)}.W^{(1)})_t + (\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)})_t$$

Assume that the processes $\sigma^{(q)} = (\sigma_t^{(q)})_{t\in[0,T]}$, $q = 1,2$, are adapted and *cadlag* and moreover for $q = 1,2$

$$\underline{\sigma}_T^{(q)} = \inf_{t\in[0,T]} \sigma_t^{(q)} > 0, \quad \overline{\sigma}_T^{(q)} = \sup_{t\in[0,T]} \sigma_t^{(q)} < \infty.$$

As we have seen in the proof of lemma 3.2.6 these conditions imply that

$$0 < h_n^{-1}\Delta_{j,n}\sigma^{(q)} < \infty, \quad q = 1,2.$$

Proposition 3.3.1. *(Barndorff-Nielsen and Shephard (2004c)) If $X_t^{(1)} = (\sigma^{(1)}.W^{(1)})_t$, $t \in [0,T]$ and $X_t^{(2)} = (\sigma^{(2)}.W^{(2)})_t$, $t \in [0,T]$ satisfy the above assumption, then*

$$v_{1,1}^{(n)}(X^{(1)}, X^{(2)})_T \xrightarrow{P} \rho \int_0^T \sigma_t^{(1)}\sigma_t^{(2)} dt$$

and

$$v_{2,2}^{(n)}(X^{(1)}, X^{(2)})_T \xrightarrow{P} (2\rho^2+1) \int_0^T (\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt$$

Proof. We can write

$$v_{1,1}^{(n)}(X^{(1)},X^{(2)})_T = \sum_{j=1}^{n}(\Delta_{j,n}X^{(1)})(\Delta_{j,n}X^{(2)}) =$$

$$\sum_{j=1}^{n}[\Delta_{j,n}(\sigma^{(1)}.W^{(1)})][\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)}) + \Delta_{j,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)})] =$$

$$\sum_{j=1}^{n}\Delta_{j,n}(\sigma^{(1)}.W^{(1)})\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)}) + \sum_{j=1}^{n}\Delta_{j,n}(\sigma^{(1)}.W^{(1)})\Delta_{j,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)}) =$$

$$v_{1,1}^{(n)}((\sigma^{(1)}.W^{(1)}),\rho\sigma^{(2)}.W^{(1)}))_T + v_{1,1}^{(n)}((\sigma^{(1)}.W^{(1)}),(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)}))_T \xrightarrow{P} \rho\int_0^T \sigma_t^{(1)}\sigma_t^{(2)}dt,$$

having used lemma 3.2.7 and remark 3.2.3 with $r,l=1$. Analogously, we have

$$v_{2,2}^{(n)}(X^{(1)},X^{(2)})_T = h_n^{-1}\sum_{j=1}^{n}(\Delta_{j,n}X^{(1)})^2(\Delta_{j,n}X^{(2)})^2 =$$

$$h_n^{-1}\sum_{j=1}^{n}[\Delta_{j,n}(\sigma^{(1)}.W^{(1)})]^2[\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)}) + \Delta_{j,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)})]^2 =$$

$$h_n^{-1}\sum_{j=1}^{n}[\Delta_{j,n}(\sigma^{(1)}.W^{(1)})]^2[\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)}]^2+$$

$$2h_n^{-1}\sum_{j=1}^{n}[\Delta_{j,n}(\sigma^{(1)}.W^{(1)})]^2\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)})\Delta_{j,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)})+$$

$$h_n^{-1}\sum_{j=1}^{n}[\Delta_{j,n}(\sigma^{(1)}.W^{(1)})]^2[\Delta_{j,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)})]^2.$$

Now,

$$h_n^{-1}\sum_{j=1}^{n}[\Delta_{j,n}(\sigma^{(1)}.W^{(1)})]^2[\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)}]^2 =$$

$$v_{2,2}^{(n)}((\sigma^{(1)}.W^{(1)}),(\rho\sigma^{(2)}.W^{(1)}))_T \xrightarrow{P} \mu_{1,4}\rho^2\int_0^T(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt = 3\rho^2\int_0^T(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt,$$

and

$$h_n^{-1}\sum_{j=1}^{n}[\Delta_{j,n}(\sigma^{(1)}.W^{(1)})]^2[\Delta_{j,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)})]^2 =$$

$$v_{2,2}^{(n)}((\sigma^{(1)}.W^{(1)}),(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)}))_T \xrightarrow{P} \int_0^T(1-\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt,$$

while the second sum tends to zero because $W^{(1)}$ appears with an odd power, as observed afterwards proposition 3.2.7. We conclude that

$$v_{2,2}^{(n)}(X^{(1)},X^{(2)})_T \xrightarrow{P} (2\rho^2+1)\int_0^T(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt.$$

•

To show the next proposition, which will be intensively used in chapter 4, we need a classical limit theorem concerning triangular array of random variables with finite variances.

Theorem 3.3.2. *(Loeve, 1977) Let* $\{H_{nj},\ j=1,....,j_n,\ n=1,2,....\}$ *be a double array of r.v.s independent in each row having the same distribution of a r.v.* H *with zero mean and finite variance. Define*

$$X_{nj} = a_{nj} H_{nj}, \qquad j=1,....,j_n,\ n=1,2,....$$

where a_{nj} *are real numbers. If the following conditions are satisfied*

1. $\sum_{j=1}^{j_n} P(|H| > \frac{\epsilon}{a_{nj}}) \to 0, \quad \forall \epsilon > 0;$
2. $\sum_{j=1}^{j_n} a_{nj} E(H 1_{\{|H|<\frac{\gamma}{a_{nj}}\}}) \to 0, \quad \forall \gamma > 0;$
3. $\sum_{j=1}^{j_n} a_{nj}^2 [E(H^2 1_{\{|H|<\frac{\gamma}{a_{nj}}\}}) - (E(H 1_{\{|H|<\frac{\gamma}{a_{nj}}\}}))^2] \to 0.$

then, as $n \to \infty$

$$\sum_{j=1}^{j_n} X_{nj} \xrightarrow{P} 0.$$

•

Proposition 3.3.3. *Conditions 1.,2., 3. of theorem 3.3.2 are satisfied if*

1. $\max_{j=1,....,j_n} a_{nj} \to 0;$
2. $j_n P(|H| > \frac{\epsilon}{\max_{j=1,....,j_n} a_{nj}}) \to 0, \quad \forall \epsilon > 0;$
3. $\sup_n \sum_{j=1}^{j_n} a_{nj} < \infty.$

•

Proposition 3.3.4. *(Barndorff-Nielsen and Shephard (2004c)) If the conditions of proposition 3.3.1 are held, if* $\sigma^{(q)} \perp W^{(1)}, W^{(3)}$ *for* $q=1,2$, *and* $\sigma^{(1)} \perp \sigma^{(2)}$ *and if we define*

$$w^{(n)}(X^{(1)}, X^{(2)})_T := h_n^{-1} \sum_{j=1}^{n-1} \Delta_{j,n} X^{(1)} \Delta_{j,n} X^{(2)} \Delta_{j+1,n} X^{(1)} \Delta_{j+1,n} X^{(2)}$$

then

$$w^{(n)}(X^{(1)}, X^{(2)})_T \xrightarrow{P} \int_0^T \rho^2 (\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt.$$

Proof. Since $X_t^{(1)} = (\sigma^{(1)}.W^{(1)})_t$ and $X_t^{(2)} = (\rho\sigma^{(2)}.W^{(1)})_t + (\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)})_t$ we can write

$$w^{(n)}(X^{(1)}, X^{(2)})_T := h_n^{-1} \sum_{j=1}^{n-1} \Delta_{j,n} X^{(1)} \Delta_{j,n} X^{(2)} \Delta_{j+1,n} X^{(1)} \Delta_{j+1,n} X^{(2)} =$$

$$h_n^{-1} \sum_{j=1}^{n-1} \{\Delta_{j,n}(\sigma^{(1)}.W^{(1)})[\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)}) + \Delta_{j,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)})]\times$$

$$\times \Delta_{j+1,n}(\sigma^{(1)}.W^{(1)})[\Delta_{j+1,n}(\rho\sigma^{(2)}.W^{(1)}) + \Delta_{j+1,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)})]\} =$$

$$h_n^{-1} \sum_{j=1}^{n-1} \Delta_{j,n}(\sigma^{(1)}.W^{(1)})\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)})\Delta_{j+1,n}(\sigma^{(1)}.W^{(1)})\Delta_{j+1,n}(\rho\sigma^{(2)}.W^{(1)}) +$$

$$h_n^{-1} \sum_{j=1}^{n-1} \Delta_{j,n}(\sigma^{(1)}.W^{(1)})\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)})\Delta_{j+1,n}(\sigma^{(1)}.W^{(1)})\Delta_{j+1,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)}) +$$

$$h_n^{-1} \sum_{j=1}^{n-1} \Delta_{j,n}(\sigma^{(1)}.W^{(1)})\Delta_{j,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)})\Delta_{j+1,n}(\sigma^{(1)}.W^{(1)})\Delta_{j+1,n}(\rho\sigma^{(2)}.W^{(1)}) +$$

$$h_n^{-1}\sum_{j=1}^{n-1}\Delta_{j,n}(\sigma^{(1)}.W^{(1)})\Delta_{j,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)})\Delta_{j+1,n}(\sigma^{(1)}.W^{(1)})\Delta_{j+1,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)}).$$

We have to study the limit in probability of each term. Our objective is to use theorem 3.3.2 and therefore we show that the conditions of proposition 3.3.3 are satisfied. We begin by the first term, the other being similar. Since, conditionally on $\sigma^{(1)}$, $\Delta_{j,n}(\sigma^{(1)}.W^{(1)})=\int_{t_{j-1,n}}^{t_{j,n}}\sigma_t^{(1)}dW_t^{(1)}\sim N(0,\sqrt{\int_{t_{j-1,n}}^{t_{j,n}}(\sigma_t^{(1)})^2dt})=N(0,\sqrt{\Delta_{j,n}(\sigma^{(1)})^2})$ we have $\Delta_{j,n}(\sigma^{(1)}.W^{(1)})\overset{D}{=}\sqrt{\Delta_{j,n}(\sigma^{(1)})^2}Z_{j,n}^{(1)}$ where $Z_{j,n}^{(1)}$, $j=1,....,n-1$ are independent standard Gaussian r.v.s, and the same holds for the others factors, we can write

$$h_n^{-1}\sum_{j=1}^{n-1}\Delta_{j,n}(\sigma^{(1)}.W^{(1)})\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)})\Delta_{j+1,n}(\sigma^{(1)}.W^{(1)})\Delta_{j+1,n}(\rho\sigma^{(2)}.W^{(1)})\overset{D}{=}$$

$$h_n^{-1}\sum_{j=1}^{n-1}(\Delta_{j,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j,n}\rho^2\sigma^{(2)})^2)^{1/2}(\Delta_{j+1,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j+1,n}\rho^2(\sigma^{(2)})^2)^{1/2}(Z_{j,n}^{(1)})^2(Z_{j+1,n}^{(1)})^2.$$

Now, $H_{nj}=(Z_{j,n}^{(1)})^2(Z_{j+1,n}^{(1)})^2$ are not independent and so we cannot directly apply theorem 3.3.2 and so we have to reason in a different way. Since it is possible to write

$$h_n^{-1}\sum_{j=1}^{n-1}\Delta_{j,n}(\sigma^{(1)}.W^{(1)})\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)})\Delta_{j+1,n}(\sigma^{(1)}.W^{(1)})\Delta_{j+1,n}(\rho\sigma^{(2)}.W^{(1)})+$$

$$-h_n^{-1}\sum_{j=1}^{n-1}(\Delta_{j,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j,n}\rho^2(\sigma^{(2)})^2)^{1/2}(\Delta_{j+1,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j+1,n}\rho^2(\sigma^{(2)})^2)^{1/2}\overset{D}{=}$$

$$h_n^{-1}\sum_{j=1}^{n-1}(\Delta_{j,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j,n}\rho^2(\sigma^{(2)})^2)^{1/2}(\Delta_{j+1,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j+1,n}\rho^2(\sigma^{(2)})^2)^{1/2}(H_{nj}-1),$$

if we are able to prove that

$$h_n^{-1}\sum_{j=1}^{n-1}(\Delta_{j,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j,n}\rho^2(\sigma^{(2)})^2)^{1/2}(\Delta_{j+1,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j+1,n}\rho^2(\sigma^{(2)})^2)^{1/2}(H_{nj}-1)\xrightarrow{P}0,$$

we can say that

$$h_n^{-1}\sum_{j=1}^{n-1}\Delta_{j,n}(\sigma^{(1)}.W^{(1)})\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)})\Delta_{j+1,n}(\sigma^{(1)}.W^{(1)})\Delta_{j+1,n}(\rho\sigma^{(2)}.W^{(1)}) \tag{3.1}$$

has the same limit in probability of

$$h_n^{-1}\sum_{j=1}^{n-1}(\Delta_{j,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j,n}\rho^2(\sigma^{(2)})^2)^{1/2}(\Delta_{j+1,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j+1,n}\rho^2(\sigma^{(2)})^2)^{1/2}$$

which is $\int_0^T\rho^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt$, applying lemma 3.2.6 and lemma 3.2.5 with $\varphi_{1t}=(\sigma_t^{(1)})^2$ and $\varphi_{2t}=\rho^2(\sigma_t^{(2)})^2$. We want to apply theorem 3.3.2 to (3.1). Let

$$b_{nj}=h_n^{-1}(\Delta_{j,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j,n}\rho^2(\sigma^{(2)})^2)^{1/2}(\Delta_{j+1,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j+1,n}\rho^2(\sigma^{(2)})^2)^{1/2}(H_{nj}-1).$$

We can say that $b_{nj}\perp b_{n,j+m}$ if $|m|>1$. Now, let $b_{nj}^{'}$ be an independent copy of b_{nj} and consider the sum

$$\sum_{j=1}^{n-1}b_{nj}+\sum_{j=1}^{n-1}b_{nj}^{'}.$$

It can be rewritten in such a way that the summands are independent as we like. In fact

$$\sum_{j=1}^{n-1} b_{nj} + \sum_{j=1}^{n-1} b_{nj}^{'} = (b_{n1} + b_{n2}^{'} + b_{n3} + b_{n4}^{'} + \cdots) + (b_{n1}^{'} + b_{n2} + b_{n3}^{'} + b_{n4} + \cdots) = S_n + S_n^{'},$$

and now each sum S_n and $S_n^{'}$ has independent summands. Hence, if we prove that $S_n \xrightarrow{P} 0$ (and $S_n^{'} \xrightarrow{P} 0$), we can conclude that $\sum_{j=1}^{n-1} b_{nj} + \sum_{j=1}^{n-1} b_{nj}^{'} \xrightarrow{P} 0$ and since $\sum_{j=1}^{n-1} b_{nj} \perp \sum_{j=1}^{n-1} b_{nj}^{'}$, then

$$\sum_{j=1}^{n-1} b_{nj} \xrightarrow{P} 0,$$

as required. However, the conclusion $S_n \xrightarrow{P} 0$ (and $S_n^{'} \xrightarrow{P} 0$) follows by theorem 3.3.2 because $Eb_{nj} = Eb_{nj}^{'} = 0$, since $E(Z_{j,n}^{(1)})^2(Z_{j+1,n}^{(1)})^2 = 1$ and it has finite variance, $b_{nj} = a_{nj}(H_{nj} - 1)$ and a_{nj} satisfies conditions 1, 2 and 3 of proposition 3.3.3. In fact, we have

$$a_{nj} = h_n^{-1}(\Delta_{j,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j,n}\rho^2(\sigma^{(2)})^2)^{1/2}(\Delta_{j+1,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j+1,n}\rho^2(\sigma^{(2)})^2)^{1/2}.$$

We have to verify conditions of proposition 3.3.3, taking into account that under conditioning the terms a_{nj} are deterministic. By hypothesis $h_n^{-1}(\Delta_{j,n}(\sigma^{(1)})^2)^{1/2}$ is bounded uniformly in j and all the other factors tend to zero, so that

$$\max_{1\le j\le n-1} a_{nj} \le$$

$$h_n^{-1}[h_n(\overline{\sigma}^{(1)})^2]^{1/2}|\rho|[h_n(\overline{\sigma}^{(2)})^2]^{1/2}[h_n(\overline{\sigma}^{(1)})^2]^{1/2}|\rho|[h_n(\overline{\sigma}^{(2)})^2]^{1/2} =$$

$$h_n\rho^2(\overline{\sigma}^{(1)})^2(\overline{\sigma}^{(2)})^2 \to 0.$$

In particular $\max_{1\le j\le n-1} a_{nj} = O(h_n)$. Moreover, it is easy to see that $nP(|H| > \frac{\epsilon}{\max_{1\le j\le n-1} a_{nj}})$ $\to 0$ since

$$nP(|H| > \frac{\epsilon}{\max_{1\le j\le n-1} a_{nj}}) \le \frac{nE|H|^2(\max_{1\le j\le n-1} a_{nj})^2}{\epsilon^2} = \frac{nE|H|^2 O(h_n^2)}{\epsilon^2} \to 0, \quad \forall \epsilon > 0,$$

Finally

$$\sup_n \sum_{j=1}^{n-1} a_{nj} =$$

$$\sup_n \sum_{j=1}^{n-1} h_n^{-1}(\Delta_{j,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j,n}\rho^2(\sigma^{(2)})^2)^{1/2}(\Delta_{j+1,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j+1,n}\rho^2(\sigma^{(2)})^2)^{1/2} \le$$

$$\sup_n \sum_{j=1}^{n-1} h_n^{-1}[h_n(\overline{\sigma}^{(1)})^2]^{1/2}|\rho|[h_n(\overline{\sigma}^{(2)})^2]^{1/2}[h_n(\overline{\sigma}^{(1)})^2]^{1/2}|\rho|[h_n(\overline{\sigma}^{(2)})^2]^{1/2} =$$

$$nh_n\rho^2(\overline{\sigma}^{(1)})^2(\overline{\sigma}^{(2)})^2 = T\rho^2(\overline{\sigma}^{(1)})^2(\overline{\sigma}^{(2)})^2 < \infty, \quad P-a.s.,$$

as required.

The same reasoning can be applied to the other terms. For example, take the second one. We see that

$$h_n^{-1}\sum_{j=1}^{n-1} \Delta_{j,n}(\sigma^{(1)}.W^{(1)})\Delta_{j,n}(\rho\sigma^{(2)}.W^{(1)})\Delta_{j+1,n}(\sigma^{(1)}.W^{(1)})\Delta_{j+1,n}(\sigma^{(2)}\sqrt{1-\rho^2}.W^{(3)}) \stackrel{D}{=}$$

$$h_n^{-1}\sum_{j=1}^{n-1}(\Delta_{j,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j,n}\rho^2(\sigma^{(2)})^2)^{1/2}(\Delta_{j+1,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j+1,n}(1-\rho^2)(\sigma^{(2)})^2)^{1/2}H_{nj}.$$

where $H_{nj} = (Z_{j,n}^{(1)})^2 Z_{j+1,n}^{(1)} Z_{j+1,n}^{(2)}$. Now, since $\{H_{nj} = (Z_{j,n}^{(1)})^2 Z_{j+1,n}^{(1)} Z_{j+1,n}^{(2)}, j = 1, 2, ..., n-1, n = 2, 3, ...\}$ is a double array of random variables having the same distribution with $EH_{nj} = E[(Z_{j,n}^{(1)})^2 Z_{j+1,n}^{(1)}]E[Z_{j+1,n}^{(2)}] = 0$ and finite variance, we can proceed in the same way setting directly

$$b_{nj} =$$

$$h_n^{-1}\sum_{j=1}^{n-1}(\Delta_{j,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j,n}\rho^2(\sigma^{(2)})^2)^{1/2}(\Delta_{j+1,n}(\sigma^{(1)})^2)^{1/2}(\Delta_{j+1,n}(1-\rho^2)(\sigma^{(2)})^2)^{1/2} Z_{j,n}^{(1)})^2 Z_{j+1,n}^{(1)} Z_{j+1,n}^{(2)}.$$

Then, the limit in probability is zero as well as for the third and the fourth terms and the proposition is proved.

•

Proposition 3.3.5. *(Central Limit Theorem) (Barndorff-Nielsen and Shephard (2004c)) If the conditions of theorem 3.3.1 hold, then*

$$\frac{h^{-1/2}(v_{1,1}^{(n)}(X^{(1)}, X^{(2)})_T - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)} dt)}{\sqrt{v_{2,2}^{(n)}(X^{(1)}, X^{(2)})_T - w^{(n)}(X^{(1)}, X^{(2)})_T}} \xrightarrow{D} N(0,1),$$

as $n \to \infty$.

•

3.4 The presence of the drift term

Generally, the diffusion part of a stock price model presents a drift term of the type $\int_0^t a_t dt$ where $a = (a_t)_{t\in[0,T]}$ is a stochastic process which satisfies specified assumptions. In this section we want to show that the drift term is in fact negligible and does not affect the results of this chapter. In particular, we assume that

$$\max_{1\le j\le n} |a_{t_{j,n}}^{(q)} - a_{t_{j-1,n}}^{(q)}| = O(h_n), \quad pathwise,$$

$q = 1, 2$. Remark that this assumption is stronger than continuity and moreover it is satisfied by a pathwise Lipschitz function. Since the time interval $[0, T]$ is compact, the process $a^{(q)}$ is also $P-a.s.$bounded, $\overline{a^{(q)}}_T = \max_{0\le t\le T} a_t < \infty$. Now, define $D^{(1)}$ and $D^{(2)}$ in the following way

$$D_t^{(1)} = \int_0^t a_s^{(1)} ds + \int_0^t \sigma_s^{(1)} dW_s^{(1)} = \int_0^t a_s^{(1)} ds + X_t^{(1)}$$

$$D_t^{(2)} = \int_0^t a_s^{(2)} ds + \int_0^t \rho\sigma_s^{(2)} dW_s^{(1)} + \int_0^t \sqrt{1-\rho^2}\sigma_s^{(2)} dW_s^{(3)} = \int_0^t a_s^{(2)} ds + X_t^{(2)}$$

We show that

$$|v_{1,1}^{(n)}(D^{(1)}, D^{(2)})_T - v_{1,1}^{(n)}(X^{(1)}, X^{(2)})_T| \to 0, \quad P-a.s.,$$

$$|v_{2,2}^{(n)}(D^{(1)}, D^{(2)})_T - v_{2,2}^{(n)}(X^{(1)}, X^{(2)})_T| \to 0, \quad P-a.s.,$$

$$|w^{(n)}(D^{(1)}, D^{(2)})_T - w^{(n)}(X^{(1)}, X^{(2)})_T| \to 0, \quad P-a.s.,$$

which imply that the drift term is negligible in proposition 3.3.1, in proposition 3.3.4 and in proposition 3.3.5. We have

$$|v_{1,1}^{(n)}(D^{(1)}, D^{(2)})_T - v_{1,1}^{(n)}(X^{(1)}, X^{(2)})_T| = |\sum_{j=1}^n (\Delta_{j,n}D^{(1)}\Delta_{j,n}D^{(2)} - \Delta_{j,n}X^{(1)}\Delta_{j,n}X^{(2)})| =$$

$$|\sum_{j=1}^n [(\Delta_{j,n}a^{(1)} + \Delta_{j,n}X^{(1)})(\Delta_{j,n}a^{(2)} + \Delta_{j,n}X^{(2)}) - \Delta_{j,n}X^{(1)}\Delta_{j,n}X^{(2)}]| \le$$

$$|\sum_{j=1}^{n}\Delta_{j,n}a^{(1)}\Delta_{j,n}a^{(2)}|+|\sum_{j=1}^{n}\Delta_{j,n}a^{(1)}\Delta_{j,n}X^{(2)}|+|\sum_{j=1}^{n}\Delta_{j,n}X^{(1)}\Delta_{j,n}a^{(2)}|.$$

Since by the Lévy modulus of continuity and Mancini (2004)

$$\sup_j \frac{|\Delta_{j,n}X^{(q)}|}{\sqrt{2h_n log\frac{1}{h_n}}}\le K_q(\omega)<\infty,\quad q=1,2,$$

where $K_1(\omega)$ and $K_2(\omega)$ are appropriate constants, we can write

$$|\sum_{j=1}^{n}\Delta_{j,n}a^{(1)}\Delta_{j,n}a^{(2)}|\le|nh_n\overline{a^{(1)}}_T h_n\overline{a^{(2)}}_T|=Th_n|\overline{a^{(1)}}_T||\overline{a^{(2)}}_T|=O(h_n),\quad P-a.s.,$$

while

$$|\sum_{j=1}^{n}\Delta_{j,n}a^{(1)}\Delta_{j,n}X^{(2)}|\le K_2(\omega)nh_n\sqrt{2h_n log\frac{1}{h_n}}|\overline{a^{(1)}}_T|=$$

$$TK_2(\omega)\sqrt{2h_n log\frac{1}{h_n}}|\overline{a^{(1)}}_T|=O(\sqrt{2h_n log\frac{1}{h_n}}),$$

as required. Moreover

$$|v_{2,2}^{(n)}(D^{(1)},D^{(2)})_T-v_{2,2}^{(n)}(X^{(1)},X^{(2)})_T|=$$

$$|h_n^{-1}\sum_{j=1}^{n}(\Delta_{j,n}D^{(1)})^2(\Delta_{j,n}D^{(2)})^2-h_n^{-1}\sum_{j=1}^{n}(\Delta_{j,n}X^{(1)})^2(\Delta_{j,n}X^{(2)})^2|\le$$

$$|h_n^{-1}\sum_{j=1}^{n}(\Delta_{j,n}a^{(1)})^2(\Delta_{j,n}a^{(2)})^2|+|h_n^{-1}\sum_{j=1}^{n}(\Delta_{j,n}a^{(1)})^2(\Delta_{j,n}X^{(2)})^2|+$$

$$|h_n^{-1}\sum_{j=1}^{n}2(\Delta_{j,n}a^{(1)})^2(\Delta_{j,n}a^{(2)})(\Delta_{j,n}X^{(2)})|+|h_n^{-1}\sum_{j=1}^{n}(\Delta_{j,n}X^{(1)})^2(\Delta_{j,n}a^{(2)})^2|+$$

$$|h_n^{-1}\sum_{j=1}^{n}2(\Delta_{j,n}X^{(1)})^2(\Delta_{j,n}a^{(2)})(\Delta_{j,n}X^{(2)})|+|h_n^{-1}\sum_{j=1}^{n}2(\Delta_{j,n}a^{(1)})(\Delta_{j,n}X^{(1)})(\Delta_{j,n}a^{(2)})^2|+$$

$$|h_n^{-1}\sum_{j=1}^{n}2(\Delta_{j,n}a^{(1)})(\Delta_{j,n}X^{(1)})(\Delta_{j,n}X^{(2)})^2|+|h_n^{-1}\sum_{j=1}^{n}4(\Delta_{j,n}a^{(1)})(\Delta_{j,n}a^{(2)})(\Delta_{j,n}X^{(1)})(\Delta_{j,n}X^{(2)})|,$$

and the eight sums tend to zero. For example

$$h_n^{-1}\sum_{j=1}^{n}(\Delta_{j,n}a^{(2)})^2(\Delta_{j,n}X^{(1)})^2\le h_n^{-1}nK_1^2(\omega)(h_n log\frac{1}{h_n})h_n^2(\overline{a^{(1)}}_T)^2=$$

$$TK_1^2(\omega)(h_n log\frac{1}{h_n})(\overline{a^{(1)}}_T)^2=O(h_n log\frac{1}{h_n}),$$

and

$$h_n^{-1}\sum_{j=1}^{n}2(\Delta_{j,n}X^{(1)})^2(\Delta_{j,n}a^{(2)})(\Delta_{j,n}X^{(2)})\le h_n^{-1}nK_1^2(\omega)K_2(\omega)(h_n log\frac{1}{h_n})^{3/2}h_n(\overline{a^{(2)}}_T)^2=$$

$$TK_1^2(\omega)K_2(\omega)\sqrt{h_n}(log\frac{1}{h_n})^{3/2}(\overline{a^{(1)}}_T)^2=O(\sqrt{h_n}(log\frac{1}{h_n})^{3/2}).$$

Finally, consider

$$|w^{(n)}(D^{(1)},D^{(2)})_T-w^{(n)}(X^{(1)},X^{(2)})_T|=$$

$$|h_n^{-1}\sum_{j=1}^{n-1}[\Delta_j D^{(1)}\Delta_j D^{(2)}\Delta_{j+1}D^{(1)}\Delta_{j+1}D^{(2)} - \Delta_j X^{(1)}\Delta_j X^{(2)}\Delta_{j+1}X^{(1)}\Delta_{j+1}X^{(2)}]|.$$

Here the sum is done by many terms each of which tends to zero because the assumption about the processes $a^{(q)}$ is uniform on j and then the shifted increment does not affect the convergence. In fact, for example

$$|h_n^{-1}\sum_{j=1}^{n-1}\Delta_j X^{(1)}\Delta_j X^{(2)}\Delta_{j+1}a^{(1)}\Delta_{j+1}a^{(2)}| \leq$$

$$nh_n^{-1}M_1(\omega)M_2(\omega)(h_n log\frac{1}{h_n})h_n\overline{a^{(1)}}_T h_n\overline{a^{(2)}}_T =$$

$$TM_1(\omega)M_2(\omega)(h_n log\frac{1}{h_n})\overline{a^{(1)}}_T\overline{a^{(2)}}_T = O(h_n log\frac{1}{h_n}),$$

as required.

•

Chapter 4

Main results

This chapter has been written in collaboration with Cecilia Mancini, Dipartimento di Matematica per le Decisioni, University of Florence.

"Non sarebbe possibile immaginare nulla di cosi' strano e poco credibile che non sia stato affermato da qualche filosofo."
(Rene' Descartes, Discorso sul metodo)

4.1 Introduction

In this chapter we present the main results of this work relative to the identification of the covariation between the two diffusion parts of two processes driven by stochastic volatility and jumps where the jump components are Lévy processes. We consider $dX_t^{(1)} = a_t^{(1)}dt + \sigma_t^{(1)}dW_t^{(1)} + dJ_t^{(1)}$ and $dX_t^{(2)} = a_t^{(2)}dt + \sigma_t^{(2)}dW_t^{(2)} + dJ_t^{(2)}$, for $t \in [0,T]$, where $W_t^{(2)} = \rho W_t^{(1)} + \sqrt{1-\rho^2}W_t^{(3)}$, whereas $W^{(1)} = (W_t^{(1)},\ t \in [0,T])$ and $W^{(3)} = (W_t^{(3)},\ t \in [0,T])$ are independent Wiener processes and $J^{(1)}$ and $J^{(2)}$ are pure jump Lévy processes. Such processes, $X^{(q)}$ are in fact used to model the log-price of two financial assets. A commonly used approach to estimate the correlation coefficient, ρ, between two diffusion parts is to take the sum of cross products $\sum_{j=1}^{n} \Delta_j X^{(1)} \Delta_j X^{(2)}$; however this is not correct when the processes $X^{(q)}$ contain jumps since such a sum approaches the quadratic covariation containing also the jump term. Our estimator is based on a truncation principle (Mancini, 2005) allowing to detect the presence of jumps. More precisely, it is crucial to single out the time intervals where the jumps have not occurred. That is done through an indicator function which estimates whether the process has jumped or not, depending on whether the increment $X_{t_j} - X_{t_{j-1}}$ is too big in absolute value with respect to a proper function of the length of the time interval $t_j - t_{j-1} = h$, $\forall j$. We derive an appropriate estimator of the continuous part of the covariation process $[X^{(1)}, X^{(2)}]_T$. In particular, we introduce the process $\tilde{v}_{1,1}^{(n)}(X^{(1)}, X^{(2)})_T = \sum_{j=1}^{n} \Delta_j X^{(1)} 1_{\{(\Delta_j X^{(1)})^2 \le r(h)\}} \Delta_j X^{(2)} 1_{\{(\Delta_j X^{(2)})^2 \le r(h)\}}$, which represents the (realized) quadratic covariation "weighted" by the truncation principle. We call such a process *threshold estimator*. We show the consistency of $\tilde{v}_{1,1}^{(n)}(X^{(1)}, X^{(2)})_T$ by using results of Barndorff-Nielsen and Shephard (2004), presented in chapter 3, which analyzed the same problem in absence of jumps. Moreover, after introducing a truncation version of the cross product variation of order (1,1), $h^{-1}\sum_{j=1}^{n-1}\prod_{i=0}^{1}\Delta_{j+i}X^{(1)}\prod_{i=0}^{1}\Delta_{j+i}X^{(2)}$, we show that our estimator is asymptotically normal and converges with speed $\sqrt{h}$. The observation times are deterministic equally spaced, however our results hold even when the observations are not equally spaced. The period of time $[0,T]$ is fixed and the number of observed returns is assumed to go to infinity.

4.2 Finite activity case

4.2.1 Consistency

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P)$, let $X^{(1)} = (X_t^{(2)})_{t\in[0,T]}$ and $X^{(2)} = (X_t^{(2)})_{t\in[0,T]}$ be two real processes defined by

$$X_t^{(1)} = \int_0^t a_s^{(1)} ds + \int_0^t \sigma_s^{(1)} dW_s^{(1)} + \int_0^t \gamma_s^{(1)} dN_s^{(1)}, \quad t \in [0,T] \tag{4.1}$$

$$X_t^{(2)} = \int_0^t a_s^{(2)} ds + \int_0^t \sigma_s^{(2)} dW_s^{(2)} + \int_0^t \gamma_s^{(2)} dN_s^{(2)}, \quad t \in [0,T] \tag{4.2}$$

where the diffusion parts satisfy condition of section 3.3. The jump component is a compound Poisson process

$$J_t^{(q)} = \int_0^t \gamma_s^{(q)} dN_s^{(q)} = \sum_{k=1}^{N_t^{(q)}} \gamma_{\tau_k^{(q)}}, \quad q = 1,2$$

in which $\{\tau_k^{(q)},\ k = 1, ..., N_T^{(q)}\}$ denote the instants of jumps of $J^{(q)}$, $q = 1,2$, and $\gamma_{\tau_k^{(q)}}$ denote the size of the jump occurred at $\tau_k^{(q)}$.

For completeness, we repeat the assumptions just introduced in chapter 3. Let $\pi^{[0,T]} = \{0 = t_0 < t_1 < \cdots < t_n = T\}$ be a partition of $[0,T]$ and let $\{\pi_n^{[0,T]}, n \in \mathbb{N}\}$ be a sequence of partitions of $[0,T]$ such that $\max_{1\le j\le n} |t_{j,n} - t_{j-1,n}| \to 0$ as $n \to \infty$. In this work we assume equally spaced subdivisions, i.e., $h_n := t_{j,n} - t_{j-1,n} = \frac{T}{n}$ for every $n = 1, 2, ...$ Hence $h_n \to 0$ as $n \to \infty$. Moreover, let $\Delta_{j,n} X := X_{t_{j,n}} - X_{t_{j-1,n}}$. To simplify notations denote h_n by h and $\Delta_{j,n} X$ by $\Delta_j X$. So, we assume

1. $\limsup_{h\to 0} \frac{\sup_{1\le j\le n} |\int_{t_{j-1}}^{t_j} a_s^{(q)} ds|}{\sqrt{h log \frac{1}{h}}} \le C_q(\omega) < \infty, \quad q = 1,2;$
2. $\int_0^T (\sigma_s^{(q)})^2 ds < \infty, \quad P - a.s., \quad q = 1,2;$
3. $\limsup_{h\to 0} \frac{\sup_{1\le j\le n} |\int_{t_{j-1}}^{t_j} (\sigma_s^{(q)})^2 ds|}{h} \le M_q(\omega) < \infty, \quad q = 1,2;$
4. the deterministic function $r(h)$, $h \mapsto r(h)$, satisfies the following properties: $\lim_{h\to 0} r(h) = 0$ and $\lim_{h\to 0} \frac{h log \frac{1}{h}}{r(h)} = 0$; we denote $r(h)$ by r_h.

We know that $P - a.s.$ for sufficiently small h (Mancini, 2004)

$$1_{\{(\Delta_j X^{(q)})^2 \le r_h\}} = 1_{\{\Delta_j N^{(q)} = 0\}}, \quad j = 1, 2, ..., n, \quad q = 1,2.$$

In the case where $X^{(q)}$ include a finite activity jump component, we consider **threshold estimators**

$$\tilde{v}_{1,1}^{(n)}(X^{(1)}, X^{(2)})_T = \sum_{j=1}^n \Delta_j X^{(1)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} \Delta_j X^{(2)} 1_{\{(\Delta_j X^{(2)})^2 \le r_h\}}$$

$$\tilde{v}_{2,2}^{(n)}(X^{(1)}, X^{(2)})_T = h^{-1} \sum_{j=1}^n (\Delta_j X^{(1)})^2 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} (\Delta_j X^{(2)})^2 1_{\{(\Delta_j X^{(2)})^2 \le r_h\}}$$

and

$$\tilde{w}^{(n)}(X^{(1)}, X^{(2)})_T := h^{-1} \sum_{j=1}^{n-1} [\prod_{i=0}^{1} \Delta_{j+i} X^{(1)} 1_{\{(\Delta_{j+i} X^{(1)})^2 \le r_h\}} \prod_{i=0}^{1} \Delta_{j+i} X^{(2)} 1_{\{(\Delta_{j+i} X^{(2)})^2 \le r_h\}}].$$

Theorem 4.2.1. *(Consistency) Let $(X_t^{(1)})_{t\in[0,T]}$ and $(X_t^{(2)})_{t\in[0,T]}$ two volatility processes of the form (4.1) and (4.2) and satisfying assumptions above. Then*

$$\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T \xrightarrow{P} \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt.$$

Proof. Since for small h $1_{\{(\Delta_j X^{(q)})^2\le r_h\}} = 1_{\{\Delta_j N^{(q)}=0\}}$, $q=1,2$ for every $j=1,2,....,n$, we can write

$$\begin{aligned}
\text{P}lim_h \tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T &= \text{P}lim_h \sum_{j=1}^n \Delta_j X^{(1)} 1_{\{(\Delta_j X^{(1)})^2\le r_h\}} \Delta_j X^{(2)} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}} \\
&= \text{P}lim_h \sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} 1_{\{\Delta_j N^{(1)}=0\}} \Delta_j (X^{(2)})^{(c)} 1_{\{\Delta_j N^{(2)}=0\}} \\
&= \text{P}lim_h \sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} (1-1_{\{\Delta_j N^{(1)}\neq 0\}}) \Delta_j (X^{(2)})^{(c)} (1-1_{\{\Delta_j N^{(1)}\neq 0\}})
\end{aligned}$$

where $(X^{(q)})^{(c)}$ denote the continuous component of the process. Hence we have

$$\text{P}lim_h \tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T =$$

$$\text{P}lim_h v_{1,1}^{(n)}((X^{(1)})^{(c)},(X^{(2)})^{(c)})_T - \text{P}lim_h \sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} \Delta_j (X^{(1)})^{(c)} 1_{\{\Delta_j N^{(2)}\neq 0\}} +$$

$$-\text{P}lim_h \sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} \Delta_j (X^{(2)})^{(c)} 1_{\{\Delta_j N^{(1)}\neq 0\}} +$$

$$\text{P}lim_h \sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} 1_{\{\Delta_j N^{(1)}\neq 0\}} \Delta_j (X^{(2)})^{(c)} 1_{\{\Delta_j N^{(2)}\neq 0\}}.$$

By proposition 3.3.1, the first term is the statement of the theorem, whereas all other terms are zero. For example, as for the second one

$$\text{P}lim_h \sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} \Delta_j (X^{(2)})^{(c)} 1_{\{\Delta_j N^{(2)}\neq 0\}} \le \text{P}lim_h \sum_{j=1}^n |\Delta_j (X^{(1)})^{(c)}| |\Delta_j (X^{(2)})^{(c)}| 1_{\{\Delta_j N^{(2)}\neq 0\}} \le$$

$$\text{P}lim_h \sup_j |\Delta_j (X^{(1)})^{(c)}| \sup_j |\Delta_j (X^{(1)})^{(c)}| \sum_{j=1}^n 1_{\{\Delta_j N^{(2)}\neq 0\}} \le$$

$$\text{P}lim_h \sup_j \frac{|\Delta_j (X^{(1)})^{(c)}|}{\sqrt{hlog\frac{1}{h}}} \sup_j \frac{|\Delta_j (X^{(2)})^{(c)}|}{\sqrt{hlog\frac{1}{h}}} hlog\frac{1}{h} N_T^{(2)} \le$$

$$\text{P}lim_h K_1(\omega) K_2(\omega) hlog\frac{1}{h} N_T^{(2)} = 0,$$

since by assumption and by Mancini (2004), we have $P-a.s.$

$$\sup_j \frac{|\Delta_j (X^{(1)})^{(c)}|}{\sqrt{2hlog\frac{1}{h}}} = \sup_j \frac{|\int_{t_{j-1}}^{t_j} a_s^{(1)}ds + \int_{t_{j-1}}^{t_j} \sigma_s^{(1)} dW_s^{(1)}|}{\sqrt{2hlog\frac{1}{h}}} < K_1(\omega) < \infty$$

and $P-a.s.$

$$\sup_j \frac{|\Delta_j (X^{(2)})^{(c)}|}{\sqrt{2hlog\frac{1}{h}}} = \sup_j \frac{|\int_{t_{j-1}}^{t_j} a_s^{(2)}ds + \int_{t_{j-1}}^{t_j} \rho\sigma_s^{(2)} dW_s^{(1)} + \int_{t_{j-1}}^{t_j} \sqrt{1-\rho^2}\sigma_s^{(2)} dW_s^{(3)}|}{\sqrt{2hlog\frac{1}{h}}} < K_2(\omega) < \infty.$$

This concludes the proof of the theorem.

•

Theorem 4.2.2. *If the same conditions of theorem 4.2.1 hold, then*

$$\tilde{v}_{2,2}^{(n)}(X^{(1)},X^{(2)})_T-\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T\xrightarrow{P}\int_0^T(1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt.$$

Proof. We will prove that

$$\tilde{v}_{2,2}^{(n)}(X^{(1)},X^{(2)})_T\xrightarrow{P}\int_0^T(2\rho^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt$$

and

$$\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T\xrightarrow{P}\int_0^T\rho^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt.$$

As in theorem 4.2.1 we can write

$$\mathrm{P}lim_h\tilde{v}_{2,2}^{(n)}(X^{(1)},X^{(2)})_T=\mathrm{P}lim_hh^{-1}\sum_{j=1}^n(\Delta_jX^{(1)})^21_{\{(\Delta_jX^{(1)})^2\le r_h\}}(\Delta_jX_2)^21_{\{(\Delta_jX_2)^2\le r_h\}}=$$

$$\mathrm{P}lim_hh^{-1}\sum_{j=1}^n(\Delta_j(X^{(1)})^{(c)})^21_{\{\Delta_jN^{(1)}=0\}}(\Delta_j(X_2)^{(c)})^21_{\{\Delta_jN^{(2)}=0\}}=$$

$$\mathrm{P}lim_hh^{-1}\sum_{j=1}^n(\Delta_j(X^{(1)})^{(c)})^2(1-1_{\{\Delta_jN^{(1)}\neq0\}})(\Delta_j(X_2)^{(c)})^2(1-1_{\{\Delta_jN^{(2)}\neq0\}}),$$

so that

$$\mathrm{P}lim_h\tilde{v}_{2,2}^{(n)}(X^{(1)},X_2)_T=$$

$$\mathrm{P}lim_hv_{2,2}^{(n)}(X^{(1)})^{(c)},(X^{(2)})^{(c)})_T-\mathrm{P}lim_hh^{-1}\sum_{j=1}^n(\Delta_j(X^{(1)})^{(c)})^2(\Delta_j(X^{(2)})^{(c)})^21_{\{\Delta_jN^{(2)}\neq0\}}+$$

$$-\mathrm{P}lim_hh^{-1}\sum_{j=1}^n(\Delta_j(X^{(1)})^{(c)})^2(\Delta_j(X^{(2)})^{(c)})^21_{\{\Delta_jN^{(1)}\neq0\}}+$$

$$\mathrm{P}lim_hh^{-1}\sum_{j=1}^n(\Delta_j(X^{(1)})^{(c)})^21_{\{\Delta_jN^{(1)}\neq0\}}(\Delta_j(X^{(2)})^{(c)})^21_{\{\Delta_jN^{(2)}\neq0\}}$$

By proposition 3.3.1 $\mathrm{P}lim_hv_{2,2}^{(n)}((X^{(1)})^{(c)},(X^{(2)})^{(c)})_T=\int_0^T(2\rho^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt$ whereas the other terms are all zero. Indeed e.g.

$$\mathrm{P}lim_hh^{-1}\sum_{j=1}^n(\Delta_j(X^{(1)})^{(c)})^2(\Delta_j(X^{(2)})^{(c)})^21_{\{\Delta_jN^{(2)}\neq0\}}\le$$

$$\mathrm{P}lim_hh^{-1}\sup_j(\Delta_j(X^{(1)})^{(c)})^2\sup_j(\Delta_j(X^{(2)})^{(c)})^2\sum_{j=1}^n1_{\{\Delta_jN^{(2)}\neq0\}}\le$$

$$\mathrm{P}lim_hh^{-1}\sup_j(\Delta_j(X^{(1)})^{(c)})^2\sup_j(\Delta_j(X^{(2)})^{(c)})^2N_T^{(2)}=$$

$$\mathrm{P}lim_hh^{-1}\sup_j(\frac{|\Delta_j(X^{(1)})^{(c)}|}{\sqrt{hlog\frac{1}{h}}})^2\sup_j(\frac{|\Delta_j(X^{(2)})^{(c)}|}{\sqrt{hlog\frac{1}{h}}})^2h^2log^2\frac{1}{h}N_T^{(2)}\le\mathrm{P}lim_hK'(\omega)hlog^2\frac{1}{h}N_T^{(2)}=0.$$

Analogously

$$\mathrm{P}lim_h\tilde{w}^{(n)}(X^{(1)},X_2)_T=$$

$$\text{P}lim_h h^{-1}\sum_{j=1}^{n-1}[\prod_{i=0}^{1}\Delta_{j+i}X^{(1)}1_{\{(\Delta_{j+i}X^{(1)})^2\le r_h\}}\prod_{i=0}^{1}\Delta_{j+i}X^{(2)}1_{\{(\Delta_{j+i}X^{(2)})^2\le r_h\}}]=$$

$$\text{P}lim_h h^{-1}\sum_{j=1}^{n-1}[\prod_{i=0}^{1}\Delta_{j+i}(X^{(1)})^{(c)}1_{\{(\Delta_{j+i}N^{(1)}=0\}}\prod_{i=0}^{1}\Delta_{j+i}(X^{(2)})^{(c)}1_{\{(\Delta_{j+i}N^{(2)}=0\}}]=$$

$$\text{P}lim_h h^{-1}\sum_{j=1}^{n-1}[\prod_{i=0}^{1}\Delta_{j+i}(X^{(1)})^{(c)}(1-1_{\{(\Delta_{j+i}N^{(1)}\neq 0\}})\prod_{i=0}^{1}\Delta_{j+i}(X^{(2)})^{(c)}(1-1_{\{(\Delta_{j+i}N^{(2)}\neq 0\}})].$$

Thus, $\text{P}lim_h\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T=\text{P}lim_h w^{(n)}(X^{(1)},X^{(2)})_T=\int_0^T\rho^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt$ plus a finite number of terms whose limit in probability is zero. For example

$$\text{P}lim_h h^{-1}\sum_{j=1}^{n-1}[\prod_{i=0}^{1}\Delta_{j+i}(X^{(1)})^{(c)}1_{\{(\Delta_{j+i}N^{(1)}\neq 0\}}\prod_{i=0}^{1}\Delta_{j+i}(X^{(2)})^{(c)}1_{\{(\Delta_{j+i}N^{(2)}\neq 0\}}]\le$$

$$\text{P}lim_h h^{-1}\sum_{j=1}^{n-1}[\prod_{i=0}^{1}|\Delta_{j+i}(X^{(1)})^{(c)}|1_{\{(\Delta_{j+i}N^{(1)}\neq 0\}}\prod_{i=0}^{1}|\Delta_{j+i}(X^{(2)})^{(c)}|1_{\{(\Delta_{j+i}N^{(2)}\neq 0\}}]\le$$

$$\text{P}lim_h h^{-1}\prod_{i=0}^{1}\sup_j\frac{|\Delta_{j+i}(X^{(1)})^{(c)}|}{\sqrt{hlog\frac{1}{h}}}\prod_{i=0}^{1}\sup_j\frac{|\Delta_{j+i}(X^{(2)})^{(c)}|}{\sqrt{hlog\frac{1}{h}}}h^2log^2\frac{1}{h}\sum_{j=1}^{n-1}\prod_{i=0}^{1}1_{\{(\Delta_{j+i}N^{(1)}\neq 0\}}1_{\{(\Delta_{j+i}N^{(2)}\neq 0\}}\le$$

$$\text{P}lim_h h^{-1}\prod_{i=0}^{1}\sup_j\frac{|\Delta_{j+i}(X^{(1)})^{(c)}|}{\sqrt{hlog\frac{1}{h}}}\prod_{i=0}^{1}\sup_j\frac{|\Delta_{j+i}(X^{(2)})^{(c)}|}{\sqrt{hlog\frac{1}{h}}}h^2log^2\frac{1}{h}\sum_{j=1}^{n-1}1_{\{(\Delta_{j+i}N^{(1)}\neq 0\}}\le$$

$$\text{P}lim_h hlog^2\frac{1}{h}K_1^2(\omega)K_2^2(\omega)N_T^{(1)}=0$$

•

4.2.2 Central Limit Theorem

We show a central limit result relative to the threshold estimator by using proposition 3.3.5. Our purpose is to introduce a quantity whose asymptotic distribution is stable. In particular, we prove that the asymptotic law of a normalized version of $\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T$ is Standard Normal.

Theorem 4.2.3. *Suppose that the conditions of theorem 4.2.1 are held, then*

$$\frac{h^{-1/2}(\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T-\int_0^T\rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\tilde{v}_{2,2}^{(n)}(X^{(1)},X^{(2)})_T-\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}}\xrightarrow{D}N(0,1).$$

***Proof*.** Remark that

$$\text{D}lim_h\frac{h^{-1/2}(\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T-\int_0^T\rho\sigma_{1t}\sigma_t^{(2)}dt)}{\sqrt{\tilde{v}_{2,2}^{(n)}(X^{(1)},X^{(2)})_T-\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}}=\text{D}lim_h\frac{h^{-1/2}(\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T-\int_0^T\rho\sigma_t^{(1)}\sigma_t^{(1)}dt)}{\sqrt{\int_0^T(1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt}}$$

Now

$$\text{D}lim_h\frac{h^{-1/2}(\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T-\int_0^T\rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\int_0^T(1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt}}=$$

$$\text{D}lim_h\frac{h^{-1/2}(\sum_{j=1}^{n}\Delta_jX^{(1)}1_{\{(\Delta_jX^{(1)})^2\le r_h\}}\Delta_jX^{(2)}1_{\{(\Delta_jX^{(2)})^2\le r_h\}}-\int_0^T\rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\int_0^T(1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt}}=$$

$$\mathrm{D}lim_h \frac{h^{-1/2}(\sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} 1_{\{\Delta_j N^{(1)}=0\}} \Delta_j (X^{(2)})^{(c)} 1_{\{\Delta_j N^{(2)}=0\}} - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)} dt)}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt}} =$$

$$\mathrm{D}lim_h \frac{h^{-1/2}(\sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} \Delta_j (X^{(2)})^{(c)} - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)} dt)}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt}} +$$

$$-\mathrm{D}lim_h \frac{h^{-1/2}\sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} \Delta_j (X^{(2)})^{(c)} 1_{\{\Delta_j N^{(2)}\neq 0\}}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt}} +$$

$$-\mathrm{D}lim_h \frac{h^{-1/2}\sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} 1_{\{\Delta_j N^{(1)}\neq 0\}} \Delta_j (X^{(2)})^{(c)}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt}} +$$

$$\mathrm{D}lim_h \frac{h^{-1/2}\sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} 1_{\{\Delta_j N^{(1)}\neq 0\}} \Delta_j (X^{(2)})^{(c)} 1_{\{\Delta_j N^{(2)}\neq 0\}}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt}}$$

The first term is $N(0,1)$ whereas the other ones are all zero as in theorem 4.2.1, because, being the denominator a well defined finite quantity, we have

$$\mathrm{P}lim_h h^{-1/2} \sum_{j=1}^n \Delta_j (X^{(1)})^{(c)} \Delta_j (X^{(2)})^{(c)} 1_{\{\Delta_j N_2 \neq 0\}} \leq$$

$$\mathrm{P}lim_h h^{-1/2} \sup_j |\Delta_j (X^{(1)})^{(c)}| \sup_j |\Delta_j (X^{(1)})^{(c)}| \sum_{j=1}^n 1_{\{\Delta_j N^{(2)} \neq 0\}} \leq$$

$$\mathrm{P}lim_h \sup_j \frac{|\Delta_j (X^{(1)})^{(c)}|}{\sqrt{hlog\frac{1}{h}}} \sup_j \frac{|\Delta_j (X^{(2)})^{(c)}|}{\sqrt{hlog\frac{1}{h}}} \sqrt{h} log\frac{1}{h} N_T^{(2)} \leq$$

$$\mathrm{P}lim_h K_1(\omega) K_2(\omega) \sqrt{h} log\frac{1}{h} N_T^{(2)} = 0,$$

•

4.3 Infinite activity case

In this section, what we suppose is that the jump component have infinite activity, that is, $X^{(1)} = (X_t^{(1)})_{t\in[0,T]}$ and $X^{(2)} = (X_t^{(2)})_{t\in[0,T]}$ are two martingales of the form

$$X_t^{(1)} = \int_0^t a_s^{(1)} ds + \int_0^t \sigma_s^{(1)} dW_s^{(1)} + J_t^{(1)}, \quad t \in [0,T] \tag{4.3}$$

$$X_t^{(2)} = \int_0^t a_s^{(2)} ds + \int_0^t \sigma_s^{(2)} dW_s^{(2)} + J_t^{(2)}, \quad t \in [0,T] \tag{4.4}$$

where we assume that $J^{(q)}$ is a Lévy process; we then can write

$$J_t^{(q)} = J_{1t}^{(q)} + \tilde{J}_{2t}^{(q)} = \int_0^t \int_{|x|>1} x\mu^{(q)}(ds,dx) + \int_0^t \int_{|x|\leq 1} x\tilde{\mu}^{(q)}(ds,dx), \quad q = 1,2$$

in which $\mu^{(q)}$ is the Poisson random jump measure of the Lévy process $J^{(q)}$, $\tilde{\mu}^{(q)}(ds,dx) = \mu^{(q)}(ds,dx) - ds\nu^{(q)}(dx)$ where $\nu^{(q)}$ is the Lévy measure of $J^{(q)}$, each $J_1^{(q)}$ is a compound Poisson process exactly as in section 4.2. Obviously, the rest of the processes satisfy the usual conditions. More specifically, the paths of the infinite activity jump process $\tilde{J}_2^{(q)}$ jumps infinitely many times on each compact time interval because $\nu^{(q)}(\mathbb{R} - \{0\}) = \infty$. However, the behaviour of the Lévy measure around the origin determine the nature of the infinite activity of jumps. In fact, if

$\int_{|x|\leq 1} x^2\nu(dx) < \infty$ for any Lévy process, for smaller power of $|x|$ the integral could be infinite, and this means that the activity of jump is wild. A measure of the amount of activity of the Lévy process is given by the Blumenthal-Gatoor index which is precisely defined in the following

Definition 4.3.1. (**Blumenthal-Gatoor index**) *Let ν be a Lévy measure. The Blumenthal-Gatoor index is the real number $\alpha \in [0,2[$ defined by*

$$\alpha := \inf\{\delta > 0, \int_{|x|\leq 1} x^\delta \nu(dx) < \infty\}.$$

In general $\alpha \in [0,2]$. A compound Poisson process (finite activity) has $\alpha = 0$. The Variance Gamma process, which is characterized by a mild infinite activity has $\alpha = 0$. Moreover, a process with Blumenthal-Gatoor index $\alpha < 1$ has infinite activity but finite variation, whereas if $\alpha \geq 1$ it has infinite variation.

Remark 4.3.2. *We deal with Lévy measures ν such that for small ε*

$$\int_{|x|\leq\varepsilon} f(x)\nu(dx) \sim \int_{|x|\leq\varepsilon} \frac{f(x)}{|x|^{1+\alpha}}dx.$$

The most used models in practice, such as Variance Gamma process, the Normal Inverse Gaussian process and all α-stable processes belong to this class.

Now, by (Mancini, 2005) we can compute the following integral $\int_{|x|\leq\varepsilon} x^2\nu(dx)$. In fact

$$\int_{|x|\leq\varepsilon} x^2\nu(dx) \sim \int_{|x|\leq\varepsilon} \frac{x^2}{|x|^{1+\alpha}}dx =$$

$$\int_{-\varepsilon}^0 \frac{x^2}{(-x)^{1+\alpha}}dx + \int_0^\varepsilon \frac{x^2}{x^{1+\alpha}}dx = (-1)^{-1-\alpha}[\frac{x^{2-\alpha}}{2-\alpha}]_{-\varepsilon}^0 = [\frac{x^{2-\alpha}}{2-\alpha}]_0^\varepsilon = O(\varepsilon^{2-\alpha}).$$

Moreover

$$\int_{\varepsilon<|x|\leq 1} x\nu(dx) \sim \int_{\varepsilon<|x|\leq 1} \frac{x}{|x|^{1+\alpha}}dx =$$

$$(-1)^{-1-\alpha}\int_{-1}^{-\varepsilon} x^{-\alpha}dx + \int_\varepsilon^1 x^{-\alpha}dx = (-1)^{-2\alpha}(\frac{\varepsilon^{1-\alpha}}{1-\alpha} - \frac{1}{1-\alpha}) + (\frac{1}{1-\alpha} - \frac{\varepsilon^{1-\alpha}}{1-\alpha}) = O(c - c\varepsilon^{1-\alpha}).$$

4.3.1 Consistency

To prove the consistency result we need some notation. Let

$$\begin{aligned}
D_t^{(1)} &= \int_0^t a_s^{(1)}ds + \int_0^t \sigma_s^{(1)}dW_s^{(1)} \\
D_t^{(2)} &= \int_0^t a_s^{(2)}ds + \int_0^t \rho\sigma_s^{(2)}dW_s^{(1)} + \int_0^t \sqrt{1-\rho^2}\sigma_s^{(2)}dW_s^{(3)} \\
Y_t^{(1)} &= D_t^{(1)} + J_{1t}^{(1)} \\
Y_t^{(2)} &= D_t^{(2)} + J_{1t}^{(2)}
\end{aligned}$$

So $X_t^{(q)} = Y_t^{(q)} + \tilde{J}_{2t}^{(q)}$, $q = 1,2$. Since $\tilde{J}_{2t}^{(q)} \int_0^t \int_{|x|\leq 1} x\tilde{\mu}^{(q)}(dsdx) = \int_0^t \int_{|x|\leq 1} x(\mu^{(q)}(dsdx) - ds\nu^{(q)}(dx))$ we have

$$\begin{aligned}
E\tilde{J}_{2t}^{(q)} &= 0 \\
Var(\tilde{J}_{2t}^{(q)}) &= t\int_{|x|\leq 1} x^2\nu^{(q)}(dx) := t\eta_q^2(1)
\end{aligned}$$

$q = 1,2$.

Theorem 4.3.3. *(Consistency) Let $(X_t^{(1)})_{t\in[0,T]}$ and $(X_t^{(2)})_{t\in[0,T]}$ two volatility processes of the form (4.3) and (4.4) satisfying assumptions 1-4 of section 4.2.1. Then*

$$\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T \xrightarrow{P} \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt$$

as $n\to\infty$.

Proof. Firstly, we remark that since $\Delta_j X^{(q)} = \Delta_j Y^{(q)} + \Delta_j \tilde{J}_2^{(q)}$, $q=1,2$, we have

$$|\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt| =$$

$$|\sum_{j=1}^n \Delta_j X^{(1)} 1_{\{(\Delta_j X^{(1)})^2\le r_h\}} \Delta_j X^{(2)} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}} - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt| \le$$

$$|\sum_{j=1}^n \Delta_j Y^{(1)} 1_{\{(\Delta_j X^{(1)})^2\le r_h\}} \Delta_j Y^{(2)} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}} - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt| +$$

$$|\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2\le r_h\}} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}}| +$$

$$|\sum_{j=1}^n \Delta_j \tilde{J}_2^{(1)} \Delta_j Y^{(2)} 1_{\{(\Delta_j X^{(1)})^2\le r_h\}} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}}| +$$

$$|\sum_{j=1}^n \Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2\le r_h\}} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}}|,$$

so that, adding and subtracting $\sum_{j=1}^n \Delta_j Y^{(1)} 1_{\{(\Delta_j Y^{(1)})^2\le 4r_h\}} \Delta_j Y^{(2)} 1_{\{(\Delta_j Y^{(2)})^2\le 4r_h\}}$,

$$|\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt| \le$$

$$|\sum_{j=1}^n \Delta_j Y^{(1)} 1_{\{(\Delta_j Y^{(1)})^2\le 4r_h\}} \Delta_j Y^{(2)} 1_{\{(\Delta_j Y^{(2)})^2\le 4r_h\}} - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt| +$$

$$|\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j Y^{(2)} (1_{\{(\Delta_j X^{(1)})^2\le r_h\}} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}} - 1_{\{(\Delta_j Y^{(1)})^2\le 4r_h\}} 1_{\{(\Delta_j Y^{(2)})^2\le 4r_h\}})| +$$

$$|\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2\le r_h\}} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}}| +$$

$$|\sum_{j=1}^n \Delta_j \tilde{J}_2^{(1)} \Delta_j Y^{(2)} 1_{\{(\Delta_j X^{(1)})^2\le r_h\}} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}}| +$$

$$|\sum_{j=1}^n \Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2\le r_h\}} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}}|. \tag{4.5}$$

Our purpose is to prove that all terms of (4.5) tend to zero. The first one tends to zero in probability by theorem 4.2.1. As for the second one, recalling that $1_{A\cap B} = 1_A 1_B$, $1_{A\cup B} = 1_A + 1_B - 1_{A\cap B}$ and $1_A - 1_B = 1_{A\cap B^c} - 1_{A^c\cap B}$, we have

$$1_{A\cap B} - 1_{C\cap D} = [1_{A\cap B\cap C^c} + 1_{A\cap B\cap D^c} - 1_{A\cap B\cap C^c\cap D^c}] - [1_{A^c\cap C\cap D} + 1_{B^c\cap C\cap D} - 1_{A^c\cap B^c\cap C\cap D}],$$

so that

$$|\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j Y^{(2)} (1_{\{(\Delta_j X^{(1)})^2\le r_h\}} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}} - 1_{\{(\Delta_j Y^{(1)})^2\le 4r_h\}} 1_{\{(\Delta_j Y^{(2)})^2\le 4r_h\}})| =$$

$$|\sum_{j=1}^{n}\Delta_j Y^{(1)}\Delta_j Y^{(2)}(1_{\{(\Delta_j X^{(1)})^2\le r_h,(\Delta_j X^{(2)})^2\le r_h\}}-1_{\{(\Delta_j Y^{(1)})^2\le 4r_h,(\Delta_j Y^{(2)})^2\le 4r_h\}})|=$$

$$|\sum_{j=1}^{n}\Delta_j Y^{(1)}\Delta_j Y^{(2)}[(1_{\{(\Delta_j X^{(1)})^2\le r_h,(\Delta_j X^{(2)})^2\le r_h,(\Delta_j Y^{(1)})^2>4r_h\}}+$$

$$1_{\{(\Delta_j X^{(1)})^2\le r_h,(\Delta_j X^{(2)})^2\le r_h,(\Delta_j Y^{(2)})^2>4r_h\}}-1_{\{(\Delta_j X^{(1)})^2\le r_h,(\Delta_j X^{(2)})^2\le r_h,(\Delta_j Y^{(1)})^2>4r_h,(\Delta_j Y^{(2)})^2>4r_h\}})+$$

$$-(1_{\{(\Delta_j X^{(1)})^2>r_h,(\Delta_j Y^{(1)})^2\le 4r_h,(\Delta_j Y^{(2)})^2\le 4r_h\}}+1_{\{(\Delta_j X^{(2)})^2>r_h,(\Delta_j Y^{(1)})^2\le 4r_h,(\Delta_j Y^{(2)})^2\le 4r_h\}}+$$

$$-1_{\{(\Delta_j X^{(1)})^2>r_h,(\Delta_j X^{(2)})^2>r_h,(\Delta_j Y^{(1)})^2\le 4r_h,(\Delta_j Y^{(2)})^2\le 4r_h\}})]|=$$

The way in which the first three terms tend to zero in probability is similar, hence we only deal with the third one. Since $\sqrt{r_h}>|\Delta_j X^{(q)}|=|\Delta_j Y^{(q)}|-|\Delta_j \tilde{J}_2^{(q)}|$ implies $|\Delta_j \tilde{J}_2^{(q)}|>|\Delta_j Y^{(q)}|-\sqrt{r_h}>2\sqrt{r_h}-\sqrt{r_h}=\sqrt{r_h}$, we have $\{(\Delta_j X^{(q)})^2\le r_h,(\Delta_j Y^{(q)})^2>4r_h\}\subset\{|\Delta_j\tilde{J}_2^{(q)}|>\sqrt{r_h}\}$, $q=1,2$, we can write

$$\sum_{j=1}^{n}\Delta_j Y^{(1)}\Delta_j Y^{(2)}1_{\{(\Delta_j X^{(1)})^2\le r_h,(\Delta_j Y^{(1)})^2>4r_h\}}1_{\{(\Delta_j X^{(2)})^2\le r_h,(\Delta_j Y^{(2)})^2>4r_h\}}\le$$

$$\sum_{j=1}^{n}\Delta_j Y^{(1)}\Delta_j Y^{(2)}1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}=$$

$$\sum_{j=1}^{n}\Delta_j D^{(1)}\Delta_j D^{(2)}1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}+$$

$$\sum_{j=1}^{n}\Delta_j D^{(1)}\Delta_j J_1^{(2)}1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}+$$

$$\sum_{j=1}^{n}\Delta_j D^{(2)}\Delta_j J_1^{(1)}1_{\{|\Delta\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}+$$

$$\sum_{j=1}^{n}\Delta_j J_1^{(1)}\Delta_j J_1^{(2)}1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}\tag{4.6}$$

Let's show that each term of (4.6) tends to zero in probability. We have

$$\sum_{j=1}^{n}\Delta_j D^{(1)}\Delta_j D^{(2)}1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}\le$$

$$\sup_j|\Delta_j D^{(1)}|\sup_j|\Delta_j D^{(2)}|\sum_{j=1}^{n}1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}=$$

$$\sup_j\frac{|\Delta_j D^{(1)}|}{\sqrt{hlog\frac{1}{h}}}\sup_j\frac{|\Delta_j D^{(2)}|}{\sqrt{hlog\frac{1}{h}}}hlog\frac{1}{h}\sum_{j=1}^{n}1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}\le$$

$$K_1(\omega)K_2(\omega)hlog\frac{1}{h}\sum_{j=1}^{n}1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}$$

However, since $\{\Delta_j\tilde{J}_2^{(q)},\ j=1,2,...,n\}$ is a family identically distributed random variables and convergence in $\mathcal{L}_1$ implies convergence in probability, we have

$$hlog\frac{1}{h}E\sum_{j=1}^{n}1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}=hlog\frac{1}{h}\sum_{j=1}^{n}E(1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}})\le$$

$$hlog\frac{1}{h}\sum_{j=1}^{n}E1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}} = hlog\frac{1}{h}nP(|\Delta_1\tilde{J}_2^{(1)}|>\sqrt{r_h}) \leq$$

$$hlog\frac{1}{h}n\frac{E|\Delta_1\tilde{J}_2^{(1)}|^2}{r_h} = hlog\frac{1}{h}\frac{nh}{r_h}\eta_1^2(1) = T\frac{hlog\frac{1}{h}}{r_h}\eta_1^2(1) \to 0,$$

as required. As for the second term of (4.6), we have

$$|\sum_{j=1}^{n}\Delta_j D^{(1)}\Delta_j J_1^{(2)}1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}| \leq$$

$$\sup_j\frac{|\Delta_j D^{(1)}|}{\sqrt{hlog\frac{1}{h}}}\sqrt{hlog\frac{1}{h}}\sum_{j=1}^{n}|\Delta_j J_1^{(2)}|1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}} \leq$$

$$K_1(\omega)\sqrt{hlog\frac{1}{h}}\sum_{j=1}^{n}|\sum_{k=1}^{\Delta_j N^{(2)}}\gamma_{\tau_k^{(2)}}|1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}$$

Since $\{\Delta_j N^{(2)}\neq 0, |\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}, |\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}$ for some $j=1,2,...,n$ is implied by

$$\sum_{j=1}^{n}|\sum_{k=1}^{\Delta_j N^{(2)}}\gamma_{\tau_k^{(2)}}|1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}\neq 0,$$

we get

$$P\left(\sum_{j=1}^{n}|\sum_{k=1}^{\Delta_j N^{(2)}}\gamma_{\tau_k^{(2)}}|1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}\neq 0\right) \leq$$

$$P\left(\bigcup_{j=1}^{n}\{\Delta_j N^{(2)}\neq 0, |\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}, |\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}\right) \leq$$

$$\sum_{j=1}^{n}P(\Delta_j N^{(2)}\neq 0, |\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}, |\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}) \leq$$

$$\sum_{j=1}^{n}P(\Delta_j N^{(2)}\neq 0, |\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}) = nP(\Delta_1 N^{(2)}\neq 0)P(|\Delta_1\tilde{J}_2^{(2)}|>\sqrt{r_h}),$$

because $J_1^{(2)}$ is independent of $N^{(2)}$ being $J^{(2)}$ is a Lévy process. The last term tends to zero since $N^{(2)}$ is a Poisson process then $P(\Delta_1 N^{(2)}\neq 0) = O(h)$, whereas $P(|\Delta_1\tilde{J}_2^{(2)}|>\sqrt{r_h})$ is dominated by $\frac{h\eta_2^2(1)}{r_h}$, i.e.

$$P\left(\sum_{j=1}^{n}|\sum_{k=1}^{\Delta_j N^{(2)}}\gamma_{\tau_k^{(2)}}|1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}\neq 0\right) \leq nO(h)\frac{h\eta_2^2(1)}{r_h} \to 0$$

Finally, the last term of (4.6) tends to zero; it suffices to observe that

$$P\left(\sum_{j=1}^{n}|\sum_{k=1}^{\Delta_j N^{(1)}}\gamma_{\tau_k^{(1)}}||\sum_{k=1}^{\Delta_j N^{(2)}}\gamma_{\tau_k^{(2)}}|1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}\neq 0\right) \leq$$

$$P\left(\bigcup_{j=1}^{n}\{\Delta_j N^{(1)}\neq 0, \Delta_j N^{(2)}\neq 0, |\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}, |\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}\right) \leq$$

$$nP(\Delta_1 N^{(2)}\neq 0)P(|\Delta_1\tilde{J}_2^{(2)}|>\sqrt{r_h}) \to 0$$

In order to complete the proof that the second term of (4.5) tends to zero in probability, it remains to prove that $\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j Y^{(2)} 1_{\{(\Delta_j X^{(1)})^2 > r_h, (\Delta_j Y^{(1)})^2 \leq 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 > r_h (\Delta_j Y^{(2)})^2 \leq 4r_h\}} \to 0$ in probability. Let's remark for small h

$$\Delta_j N^{(q)} < |\Delta_j J_1^{(q)}| < |\Delta_j D^{(q)}| + |\Delta_j Y^{(q)}| \leq |\Delta_j D^{(q)}| + 2\sqrt{r_h} \leq \sup_j |\Delta_j D^{(q)}| + 2\sqrt{r_h} \to 0, \quad q = 1, 2,$$

uniformly in j. Hence, for small h on $\{(\Delta_j Y^{(q)})^2 \leq 4r_h\}$ we have $\Delta_j N^{(q)} = 0$, $j = 1, ..., n$. Therefore $\{(\Delta_j X^{(q)})^2 > r_h, (\Delta_j Y^{(q)})^2 \leq 4r_h\} \subset \{(\Delta_j D^{(q)} + \Delta_j \tilde{J}_2^{(q)})^2 > r_h\} \subset \{(\Delta_j D^{(q)})^2 > \frac{r_h}{2}\} \cup \{|\Delta_j \tilde{J}_2^{(q)}| > \frac{\sqrt{r_h}}{2}\}$, $q = 1, 2$; however, since $1_{\{(\Delta_j D^{(q)})^2 > r_h\}} = 0$, $P - a.s.$, for sufficiently small h, we obtain $1_{\{(\Delta_j X^{(q)})^2 > r_h, (\Delta_j Y^{(q)})^2 \leq 4r_h\}} \leq 1_{\{|\Delta_j \tilde{J}_2^{(q)}| > \frac{\sqrt{r_h}}{2}, \Delta_j N^{(2)} = 0\}}$, $q = 1, 2$. Then

$$\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j Y^{(2)} 1_{\{(\Delta_j X^{(1)})^2 > r_h, (\Delta_j Y^{(1)})^2 \leq 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 > r_h (\Delta_j Y^{(2)})^2 \leq 4r_h\}} \leq$$

$$\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j Y^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| > \frac{\sqrt{r_h}}{2}, \Delta_j N^{(1)} = 0\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| > \frac{\sqrt{r_h}}{2}, \Delta_j N^{(2)} = 0\}} =$$

$$\sum_{j=1}^n \Delta_j D^{(1)} \Delta_j D^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| > \frac{\sqrt{r_h}}{2}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| > \frac{\sqrt{r_h}}{2}\}} \leq$$

$$\sup_j \frac{|\Delta_j D^{(1)}|}{\sqrt{hlog\frac{1}{h}}} \sup_j \frac{|\Delta_j D^{(2)}|}{\sqrt{hlog\frac{1}{h}}} hlog\frac{1}{h} \sum_{j=1}^n 1_{\{|\Delta_j \tilde{J}_2^{(1)}| > \frac{\sqrt{r_h}}{2}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| > \frac{\sqrt{r_h}}{2}\}}$$

which tends to zero in probability as before.

Now, consider the third term of (4.5), $\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \leq r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \leq r_h\}}$, the fourth one being analogous. We have

$$\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \leq r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \leq r_h\}} =$$

$$\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j X^{(1)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| \leq 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \leq 2\sqrt{r_h}\}} +$$
$$\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j X^{(1)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}} +$$
$$\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j X^{(1)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \leq 2\sqrt{r_h}\}} +$$
$$\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j X^{(1)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| \leq 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}} \tag{4.7}$$

Let's show that each term of (4.7) tends to zero in probability. Firstly, we remark that on $\{|\Delta_j X^{(q)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(q)}| \leq 2\sqrt{r_h}\}$ for small h we have

$$|\Delta_j N^{(q)}| \leq |\Delta_j J_1^{(q)}| = |\Delta_j X^{(q)} - \Delta_j D^{(q)} - \Delta_j \tilde{J}_2^{(q)}| \leq \sqrt{r_h} + |\Delta_j D^{(q)}| + |\Delta_j \tilde{J}_2^{(q)}| \leq$$
$$\sqrt{r_h} + \sup_j |\Delta_j D^{(q)}| + \sup_j |\Delta_j \tilde{J}_2^{(q)}| \to 0,$$

for $q = 1, 2$, since by Doob's inequality $E(\sup_j |\Delta_j \tilde{J}_2^{(q)}|)^2 \leq cE|\Delta_j \tilde{J}_2^{(q)}|^2 = ch\eta_2^2(1) \to 0$. Besides on $\{|\Delta_j D^{(q)} + \Delta_j \tilde{J}_2^{(q)}| \leq \sqrt{r_h}, \Delta_j N^{(q)} = 0, |\Delta_j \tilde{J}_2^{(q)}| \leq 2\sqrt{r_h}\}$ we have $|\Delta_j D^{(q)}| \leq \sqrt{r_h} + |\Delta_j \tilde{J}_2^{(q)}| \leq 3\sqrt{r_h}$, $q = 1, 2$. Therefore, as for the first term of (4.7) we get

$$\sum_{j=1}^n \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j X^{(1)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| \leq 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \leq 2\sqrt{r_h}\}} =$$

$$\sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j X^{(1)}| \le \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \le \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} =$$

$$\sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j D^{(1)}| \le 3\sqrt{r_h}, \Delta_j N^{(1)}=0, |\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j D^{(2)}| \le 3\sqrt{r_h}, \Delta_j N^{(2)}=0, |\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} =$$

$$\sum_{j=1}^{n} \Delta_j D^{(1)} 1_{\{|\Delta_j D^{(1)}| \le 3\sqrt{r_h}, \Delta_j N^{(1)}=0\}} 1_{\{|\Delta_j D^{(2)}| \le 3\sqrt{r_h}, \Delta_j N^{(2)}=0\}} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}}.$$

However, for sufficiently small h, $1_{\{|\Delta_j D^{(q)}| \le 3\sqrt{r_h}\}} = 1$, $q = 1, 2$, $P - a.s.$; so we have

$$\sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j X^{(1)}| \le \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \le \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} =$$

$$\sum_{j=1}^{n} \Delta_j D^{(1)} 1_{\{\Delta_j N^{(1)}=0\}} 1_{\{\Delta_j N^{(2)}=0\}} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} =$$

$$\sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} +$$

$$\sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{\Delta_j N^{(1)} \neq 0\}} 1_{\{\Delta_j N^{(2)} \neq 0\}} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} +$$

$$-\sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{\Delta_j N^{(1)} \neq 0\}} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} +$$

$$-\sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{\Delta_j N^{(2)} \neq 0\}} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} \tag{4.8}$$

Each term of (4.8) tends to zero. In fact, except for the first one, being $1_{\{\Delta_j N^{(1)} \neq 0, \Delta_j N^{(2)} \neq 0\}} \le 1_{\{\Delta_j N^{(1)} \neq 0\}}$ we can write

$$\sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{\Delta_j N^{(1)} \neq 0\}} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} \le$$

$$\sup_j |\Delta_j D^{(1)}| \sum_{j=1}^{n} \Delta_j \tilde{J}_2^{(2)} 1_{\{\Delta_j N^{(1)} \neq 0\}} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}}.$$

Now, $\sup_j |\Delta_j D^{(1)}| \to 0$, $P - a.s.$, whereas for the second factor, we see that

$$E \sum_{j=1}^{n} |\Delta_j \tilde{J}_2^{(2)}| 1_{\{\Delta_j N^{(1)} \neq 0\}} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} =$$

$$\sum_{j=1}^{n} E[1_{\{\Delta_1 N^{(1)} \neq 0\}} |\Delta_1 \tilde{J}_2^{(2)}| 1_{\{|\Delta_1 \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_1 \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}}] \le$$

$$nP(\Delta_1 N^{(1)} \neq 0) E[|\Delta_1 \tilde{J}_2^{(2)}| 1_{\{|\Delta_1 \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}}] \le$$

$$T\lambda_1 E|\Delta_1 \tilde{J}_2^{(2)}| \le E(N_T^{(1)}) \sqrt{E|\Delta_1 \tilde{J}_2^{(2)}|^2} = E(N_T^{(1)}) \sqrt{h\eta_2^2(1)} = O(\sqrt{h}).$$

Indeed, for the first one we remark that

$$\sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} =$$

$$\sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} - \sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}}.$$

Now,

$$\sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}} \le$$

$$K_1(\omega)\sqrt{hlog\frac{1}{h}} \sum_{j=1}^{n} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}},$$

and

$$E\sqrt{hlog\frac{1}{h}} \sum_{j=1}^{n} |\Delta_j \tilde{J}_2^{(2)}| 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}} \le n\sqrt{hlog\frac{1}{h}} \sqrt{E|\Delta_1 \tilde{J}_2^{(2)}|^2 E 1_{\{|\Delta_1 \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}}} \le$$

$$n\sqrt{hlog\frac{1}{h}} \sqrt{\frac{h^2\eta_1^4(1)}{r_h}} = T\eta_1^2(1)\sqrt{\frac{hlog\frac{1}{h}}{r_h}} \to 0.$$

Besides, (Mancini, 2005, remark 4.2)

$$Plim_h \sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} \le$$

$$Plim_h \sum_{j=1}^{n} \Delta_j D^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}} = Plim_h [D^{(1)}, Z_h^{(2)}]_T$$

$$= Plim_h < (D^{(1)})^{(c)}, (Z_h^{(2)})^{(c)} >_T + Plim_h \sum_{t \le T} \Delta D_s^{(1)} Z_{h,s}^{(2)} = 0$$

where

$$Z_{h,t}^{(q)} = \int_0^t \int_{|x| < 2\sqrt{r_h}} x(\mu^{(q)}(ds,dx) - \nu^{(q)}(dx)ds) - t\int_{3\sqrt{r_h} \le |x| \le 1} x\nu^{(q)}(dx), \quad q = 1,2.$$

and $\Delta_j Z_h^{(q)} = \Delta_j \tilde{J}_2^{(q)} 1_{\{|\Delta_j \tilde{J}_2^{(q)}| \le 2\sqrt{r_h}\}} = \Delta_j \tilde{J}_2^{(q)} 1_{\{\sum_{s:s\in]t_{j-1},t_j]} |\Delta \tilde{J}_{2,s}^{(q)}| \le 3\sqrt{r_h}\}}$ since $P-a.s.$, for sufficiently small h, uniformly in j, on $\{|\Delta_j \tilde{J}_2^{(q)}| \le 2\sqrt{r_h}\}$ one has $|\Delta \tilde{J}_{2,s}^{(q)}| \le 3\sqrt{r_h}$.

Now, we deal with the second term of (4.7). On $\{|\Delta_j X^{(q)}| \le \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(q)}| > 2\sqrt{r_h}\}$, we have $|\Delta_j Y^{(q)}| > \sqrt{r_h}$, since $2\sqrt{r_h} - |\Delta_j Y^{(q)}| < |\Delta_j \tilde{J}_2^{(q)}| - |\Delta_j Y^{(q)}| < |\Delta_j X^{(q)}| \le \sqrt{r_h}$, $q = 1,2$, so that $P-a.s.$

$$|\Delta_j J_1^{(q)}| > \frac{\sqrt{r_h}}{2} \quad or \quad |\Delta_j D^{(q)}| > \frac{\sqrt{r_h}}{2}.$$

Since for small h $P-a.s.$, $1_{\{|\Delta_j D^{(q)}| > \frac{\sqrt{r_h}}{2}\}} = 0$ and $|\Delta_j J_1^{(q)}| > \frac{\sqrt{r_h}}{2}$ implies $\Delta_j N^{(q)} \ne 0$, we have

$$P\left(\sum_{j=1}^{n} |\Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)}| 1_{\{\Delta_j N^{(1)} \ne 0, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}\}} 1_{\{\Delta_j N^{(2)} \ne 0, |\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}} \ne 0\right) \le$$

$$P\left(\bigcup_{j=1}^{n} \{\Delta_j N^{(1)} \ne 0, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}, \Delta_j N^{(2)} \ne 0, |\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}\right) \le$$

$$\sum_{j=1}^{n} P(\Delta_j N^{(1)} \ne 0, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h})$$

which tends to zero as before.

Finally, the last two terms of (4.7) can be treated simultaneously. As above

$$|\sum_{j=1}^{n} \Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{|\Delta_j X^{(1)}| \le \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \le \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \le 2\sqrt{r_h}\}}| \le$$

$$\sum_{j=1}^{n} |\Delta_j Y^{(1)} \Delta_j \tilde{J}_2^{(2)}| 1_{\{|\Delta_j X^{(1)}| \le \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}\}}$$

which tends to zero in probability analogously as before.

It remains to consider the last term of (4.5). We can write

$$|\sum_{j=1}^{n} \Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h\}}| \le$$

$$|\sum_{j=1}^{n} \Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}}| +$$

$$|\sum_{j=1}^{n} \Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 4r_h\}}| +$$

$$|\sum_{j=1}^{n} \Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 4r_h\}}| +$$

$$|\sum_{j=1}^{n} \Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}}|$$

Again the the last three terms tend to zero since, for example

$$|\sum_{j=1}^{n} \Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}}| \le$$

$$\sum_{j=1}^{n} |\Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)}| 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}} \le$$

$$\sum_{j=1}^{n} |\Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)}| 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 4r_h\}},$$

and

$$P\left(\sum_{j=1}^{n} |\Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)}| 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}} \ne 0\right) \le$$

$$P\left(\bigcup_{j=1}^{n} \{\Delta_j N^{(1)} \ne 0, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}|\right) \to 0$$

On the contrary, taking into account remark 4.2 in Mancini (2005) we conclude

$$Plim_h \sum_{j=1}^{n} |\Delta_j \tilde{J}_2^{(1)} \Delta_j \tilde{J}_2^{(2)}| 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(2)})^2 \le 4r_h\}} \le$$

$$Plim_h \sum_{j=1}^{n} |\Delta_j \tilde{J}_2^{(1)}| 1_{\{(\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}} |\Delta_j \tilde{J}_2^{(2)}| 1_{\{(\Delta_j \tilde{J}_2^{(2)})^2 \le 4r_h\}} =$$

$$Plim_h \sum_{j=1}^{n} |\Delta_j Z_h^{(1)} \Delta_j Z_h^{(2)}| \le Plim_h \sqrt{\sum_{j=1}^{n} (\Delta_j Z_h^{(1)})^2} \sqrt{\sum_{j=1}^{n} (\Delta_j Z_h^{(2)})^2} = 0$$

by Minkowski's inequality.

•

By remark 4.3.2 we may derive a result which will be useful in the proof of the next theorem. We are interesting in the value of the integrals $\int_{2\sqrt{r_h}<|x|\le 1} x\nu(dx)$ and $\int_{|x|\le 2\sqrt{r_h}} x^2\nu(dx)$. We have

$$\int_{2\sqrt{r_h}<|x|\le 1} x\nu^{(q)}(dx) \sim \int_{2\sqrt{r_h}<|x|\le 1} \frac{x}{|x|^{1+\alpha_q}}dx = O(c - cr_h^{\frac{1-\alpha_q}{2}}), \quad q = 1, 2,$$

and

$$\eta_q^2(2\sqrt{r_h}) = \int_{|x|\le 2\sqrt{r_h}} x^2\nu^{(q)}(dx) \sim \int_{|x|\le 2\sqrt{r_h}} \frac{x^2}{|x|^{1+\alpha_q}}dx = O(r_h^{1-\frac{\alpha_q}{2}}), \quad q = 1, 2.$$

Now, by (Mancini, 2005), we can write

$$\Delta_j \tilde{J}_2^{(q)} 1_{\{|\Delta_j \tilde{J}_2^{(q)}|\le 2\sqrt{r_h}\}} = \int_{t_{j-1}}^{t_j}\int_{|x|\le 2\sqrt{r_h}} x\tilde{\mu}^{(q)}(dx, dt) - \int_{t_{j-1}}^{t_j}\int_{2\sqrt{r_h}<|x|\le 1} x\nu^{(q)}(dx)dt = \Delta_j \tilde{J}_{2m}^{(q)} - \Delta_j \tilde{J}_{2c}^{(q)}$$

where $\Delta_j \tilde{J}_{2m}^{(q)}$ denotes the martingale part of $\Delta_j \tilde{J}_2^{(q)} 1_{\{|\Delta_j \tilde{J}_2^{(q)}|\le\sqrt{r_h}\}}$, while $\Delta_j \tilde{J}_{2c}^{(q)}$ denotes the compensator of the jumps bigger than $2\sqrt{r_h}$. We have

$$E(\Delta_j \tilde{J}_{2m}^{(q)})^2 = h\eta_q^2(2\sqrt{r_h}) = hO(r_h^{1-\frac{\alpha_q}{2}}),$$

and

$$E(\Delta_j \tilde{J}_{2c}^{(q)})^2 = (\Delta_j \tilde{J}_{2c}^{(q)})^2 = h^2 O(c - cr_h^{\frac{1-\alpha_q}{2}})^2.$$

Theorem 4.3.4. *Under the same assumptions of theorem 4.3.3 and if* $\frac{h\log^2\frac{1}{h}}{r_h} \to 0$, *and moreover* $\int_0^T (\sigma^{(q)})^4 dt < \infty$, *then*

$$\tilde{v}_{2,2}^{(n)}(X^{(1)}, X^{(2)})_T - \tilde{w}^{(n)}(X^{(1)}, X^{(2)})_T \xrightarrow{P} \int_0^T (1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt$$

as $n \to \infty$.

Proof. We will prove that

$$\tilde{v}_{2,2}^{(n)}(X^{(1)}, X^{(2)})_T \xrightarrow{P} \int_0^T (2\rho^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt$$

and

$$\tilde{w}^{(n)}(X^{(1)}, X^{(2)})_T \xrightarrow{P} \int_0^T \rho^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt.$$

Let's begin with the first one. As in theorem 4.3.3 we can write

$$|\tilde{v}_{2,2}^{(n)}(X^{(1)}, X^{(2)})_T - \int_0^T (2\rho^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt| =$$

$$|h^{-1}\sum_{j=1}^n (\Delta_j X^{(1)})^2 1_{\{(\Delta_j X^{(1)})^2\le r_h\}}(\Delta_j X^{(2)})^2 1_{\{(\Delta_j X^{(2)})^2\le r_h\}} - \int_0^T (2\rho^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt| \le$$

$$|h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2 1_{\{(\Delta_j X^{(1)})^2\le r_h\}}(\Delta_j Y^{(2)})^2 1_{\{(\Delta_j X^{(2)})^2\le r_h\}} - \int_0^T (2\rho^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt| +$$

$$|h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2\le r_h\}} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}}| +$$

$$|h^{-1}\sum_{j=1}^n 2(\Delta_j Y^{(1)})^2(\Delta_j Y^{(2)})(\Delta_j \tilde{J}_2^{(2)}) 1_{\{(\Delta_j X^{(1)})^2\le r_h\}} 1_{\{(\Delta_j X^{(2)})^2\le r_h\}}| +$$

$$|h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_2^{(1)})^2(\Delta_jY^{(2)})^2 1_{\{(\Delta_jX^{(1)})^2\le r_h\}}1_{\{(\Delta_jX^{(2)})^2\le r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_2^{(1)})^2(\Delta_j\tilde{J}_2^{(2)})^2 1_{\{(\Delta_jX^{(1)})^2\le r_h\}}1_{\{(\Delta_jX^{(2)})^2\le r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}2(\Delta_jY^{(2)})(\Delta_j\tilde{J}_2^{(1)})^2(\Delta_j\tilde{J}_2^{(2)})1_{\{(\Delta_jX^{(1)})^2\le r_h\}}1_{\{(\Delta_jX^{(2)})^2\le r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}2(\Delta_jY^{(1)})(\Delta_j\tilde{J}_2^{(1)})(\Delta_jY^{(2)})^2 1_{\{(\Delta_jX^{(1)})^2\le r_h\}}1_{\{(\Delta_jX^{(2)})^2\le r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}2(\Delta_jY^{(1)})(\Delta_j\tilde{J}_2^{(1)})(\Delta_j\tilde{J}_2^{(2)})^2 1_{\{(\Delta_jX^{(1)})^2\le r_h\}}1_{\{(\Delta_jX^{(2)})^2\le r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}4(\Delta_jY^{(1)})(\Delta_jY^{(2)})(\Delta_j\tilde{J}_2^{(1)})(\Delta_j\tilde{J}_2^{(2)})1_{\{(\Delta_jX^{(1)})^2\le r_h\}}1_{\{(\Delta_jX^{(2)})^2\le r_h\}}| \tag{4.9}$$

The first term of the right side of (4.9) can be split up into two parts by adding and subtracting the quantity $h^{-1}\sum_{j=1}^{n}(\Delta_jY^{(1)})^2 1_{\{(\Delta_jY^{(1)})^2\le 4r_h\}}(\Delta_jY^{(2)})^2 1_{\{(\Delta_jY^{(2)})^2\le 4r_h\}}$; we obtain

$$|h^{-1}\sum_{j=1}^{n}(\Delta_jY^{(1)})^2 1_{\{(\Delta_jX^{(1)})^2\le r_h\}}(\Delta_jX^{(2)})^2 1_{\{(\Delta_jX^{(2)})^2\le r_h\}} - \int_0^T(2\rho^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt| \le$$

$$|h^{-1}\sum_{j=1}^{n}(\Delta_jY^{(1)})^2 1_{\{(\Delta_jY^{(1)})^2\le 4r_h\}}(\Delta_jY^{(2)})^2 1_{\{(\Delta_jY^{(2)})^2\le 4r_h\}} - \int_0^T(2\rho^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt|+$$

$$|h^{-1}\sum_{j=1}^{n}(\Delta_jY^{(1)})^2(\Delta_jY^{(2)})^2(1_{\{(\Delta_jX^{(1)})^2\le r_h\}}1_{\{(\Delta_jX^{(2)})^2\le r_h\}} - 1_{\{(\Delta_jY^{(1)})^2\le 4r_h\}}1_{\{(\Delta_jY^{(2)})^2\le 4r_h\}})|$$

Now, following the same technique used in the proof of theorem 4.3.3, we only have to prove the convergence to zero (in probability) of $|h^{-1}\sum_{j=1}^{n}(\Delta_jY^{(1)})^2(\Delta_jY^{(2)})^2 1_{\{(\Delta_jX^{(1)})^2\le r_h,(\Delta_jY^{(1)})^2>r_h\}}$ $1_{\{(\Delta_jX^{(2)})^2\le r_h,(\Delta_jY^{(2)})^2>r_h\}}|$ and of $|h^{-1}\sum_{j=1}^{n}(\Delta_jY^{(1)})^2(\Delta_jY^{(2)})^2 1_{\{(\Delta_jX^{(1)})^2>r_h,(\Delta_jY^{(1)})^2\le r_h\}}$ $1_{\{(\Delta_jX^{(2)})^2>r_h,(\Delta_jY^{(2)})^2\le r_h\}}|$. For the first one we can write (see theorem 4.3.3)

$$|h^{-1}\sum_{j=1}^{n}(\Delta_jY^{(1)})^2(\Delta_jY^{(2)})^2 1_{\{(\Delta_jX^{(1)})^2\le r_h,(\Delta_jY^{(1)})^2>r_h\}}1_{\{(\Delta_jX^{(2)})^2\le r_h,(\Delta_jY^{(2)})^2>r_h\}}| \le$$

$$|h^{-1}\sum_{j=1}^{n}(\Delta_jY^{(1)})^2(\Delta_jY^{(2)})^2 1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}| \le$$

$$|h^{-1}\sum_{j=1}^{n}(\Delta_jD^{(1)})^2(\Delta_jD^{(2)})^2 1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}(\Delta_jD^{(1)})^2(\Delta_jJ_1^{(2)})^2 1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}2(\Delta_jD^{(1)})^2(\Delta_jD^{(2)})(\Delta_jJ_1^{(2)})1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}2(\Delta_jD^{(1)})(\Delta_jD^{(2)})^2(\Delta_jJ_1^{(1)})1_{\{|\Delta_j\tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|>\sqrt{r_h}\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}4(\Delta_j D^{(1)})(\Delta_j J_1^{(1)})(\Delta_j D^{(2)})(\Delta_j J_1^{(2)})1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|>\sqrt{r_h}\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}2(\Delta_j D^{(1)})(\Delta_j J_1^{(1)})(\Delta_j J_1^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|>\sqrt{r_h}\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}(\Delta_j J_1^{(1)})^2(\Delta_j D^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|>\sqrt{r_h}\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}2(\Delta_j J_1^{(1)})^2(\Delta_j D^{(2)})(\Delta_j J_1^{(2)})1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|>\sqrt{r_h}\}}|+$$

$$|h^{-1}\sum_{j=1}^{n}(\Delta_j J_1^{(1)})^2(\Delta_j J_1^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|>\sqrt{r_h}\}}|.$$

All terms tend to zero in probability. Consider for example the first and the fifth ones. We have

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2(\Delta_j D^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|>\sqrt{r_h}\}} \leq$$

$$h^{-1}\sup_j\Big(\frac{|\Delta_j D^{(1)}|}{\sqrt{hlog\frac{1}{h}}}\Big)^2\sup_j\Big(\frac{|\Delta_j D^{(2)}|}{\sqrt{hlog\frac{1}{h}}}\Big)^2 h^2 log^2\frac{1}{h}\sum_{j=1}^{n}1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|>\sqrt{r_h}\}} \leq$$

$$K_1^2(\omega)K_2^2(\omega)hlog^2\frac{1}{h}\sum_{j=1}^{n}1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|>\sqrt{r_h}\}}.$$

However

$$E|hlog^2\frac{1}{h}\sum_{j=1}^{n}1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|>\sqrt{r_h}\}}| \leq hlog^2\frac{1}{h}\sum_{j=1}^{n}E1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}} =$$

$$hlog^2\frac{1}{h}nP(|\Delta_1 \tilde{J}_2^{(1)}|>\sqrt{r_h}) \leq hlog^2\frac{1}{h}n\frac{E|\Delta_1 \tilde{J}_2^{(1)}|^2}{r_h} = T\frac{hlog^2\frac{1}{h}}{r_h}\eta_2^2(1)\to 0$$

At the same way

$$\left|h^{-1}\sum_{j=1}^{n}4(\Delta_j D^{(1)})(\Delta_j J_1^{(1)})(\Delta_j D^{(2)})(\Delta_j J_1^{(2)})1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}}\right| \leq$$

$$K_1(\omega)K_2(\omega)log\frac{1}{h}\sum_{j=1}^{n}|\Delta_j J_1^{(1)}||\Delta_j J_1^{(2)}|1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}} =$$

$$K_1(\omega)K_2(\omega)log\frac{1}{h}\sum_{j=1}^{n}\left[|\sum_{k=1}^{\Delta_j N^{(1)}}\gamma_{\tau_k^{(1)}}||\sum_{k=1}^{\Delta_j N^{(2)}}\gamma_{\tau_k^{(2)}}|1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}}\right],$$

which tends to zero in probability since

$$P\left(log\frac{1}{h}\sum_{j=1}^{n}|\sum_{k=1}^{\Delta_j N^{(1)}}\gamma_{\tau_k^{(1)}}||\sum_{k=1}^{\Delta_j N^{(2)}}\gamma_{\tau_k^{(2)}}|1_{\{|\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\}} \neq 0\right) \leq$$

$$P\left(\bigcup_{j=1}^{n}\{\Delta_j N^{(1)}\neq 0, \Delta_j N^{(2)}\neq 0, |\Delta_j \tilde{J}_2^{(1)}|>\sqrt{r_h}\right) \leq$$

$$nP(\Delta_1 N^{(1)} \neq 0)P(|\Delta_1 \tilde{J}_2^{(1)}| > \sqrt{r_h}) \to 0,$$

as in the proof of theorem 4.3.3. To conclude the proof that the first term of (4.9) tends to zero in probability it remains to show that $|h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2(\Delta_j Y^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2 > r_h, (\Delta_j Y^{(1)})^2 \leq r_h\}} 1_{\{(\Delta_j X^{(2)})^2 > r_h, (\Delta_j Y^{(2)})^2 \leq r_h\}}| \xrightarrow{P} 0$. We can write (theorem 4.3.3)

$$h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2(\Delta_j Y^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2 > r_h, (\Delta_j Y^{(1)})^2 \leq 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 > r_h, (\Delta_j Y^{(2)})^2 \leq 4r_h\}} \leq$$

$$h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2(\Delta_j Y^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}| > \sqrt{r_h}, \Delta_j N^{(1)}=0\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| > \sqrt{r_h}, \Delta_j N^{(2)}=0\}} =$$

$$h^{-1}\sum_{j=1}^n (\Delta_j D^{(1)})^2(\Delta_j D^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}| > \sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| > \sqrt{r_h}\}},$$

which tends to zero in probability like as before.

The other terms of (4.9) tend to zero in probability. We only show this for the second, the third, the sixth and the last since the techniques we use can be replicated for the others. As for the second one, we have

$$|h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2 \leq r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \leq r_h\}}| \leq$$

$$|h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j X^{(1)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| \leq 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \leq 2\sqrt{r_h}\}}|+$$

$$|h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j X^{(1)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}}|+$$

$$|h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j X^{(1)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \leq 2\sqrt{r_h}\}}|+$$

$$|h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j X^{(1)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| \leq 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}}|. \tag{4.10}$$

The last three terms tend to zero in probability in the same way. For example take the second term like as in proof of theorem 4.3.3; we can write

$$P(\sum_{j=1}^n |(\Delta_j Y^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2| 1_{\{\Delta_j N^{(1)} \neq 0, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}\}} 1_{\{\Delta_j N^{(2)} \neq 0, |\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}} \neq 0) \leq$$

$$P(\bigcup_{j=1}^n \{\Delta_j N^{(1)} \neq 0, |\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}, \Delta_j N^{(2)} \neq 0, |\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}) \leq$$

$$nP(\Delta_1 N^{(1)} \neq 0, |\Delta_1 \tilde{J}_2^{(1)}| > 2\sqrt{r_h}) \to 0$$

For the first term of (4.10) we can still follow the proof of theorem 4.3.3 and write

$$h^{-1}\sum_{j=1}^n (\Delta_j Y^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j X^{(1)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}| \leq 2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}| \leq \sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}| \leq 2\sqrt{r_h}\}} =$$

$$h^{-1}\sum_{j=1}^n (\Delta_j D^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}| \leq 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| \leq 2\sqrt{r_h}\}}+$$

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2 1_{\{\Delta_j N^{(1)}\neq 0\}}1_{\{\Delta_j N^{(2)}\neq 0\}}(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\leq 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}+$$
$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2 1_{\{\Delta_j N^{(1)}\neq 0\}}(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\leq 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}+$$
$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2 1_{\{\Delta_j N^{(2)}\neq 0\}}(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\leq 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}. \tag{4.11}$$

The last three terms of (4.11) tend to zero. For example

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2 1_{\{\Delta_j N^{(1)}\neq 0\}}1_{\{\Delta_j N^{(2)}\neq 0\}}(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\leq 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}\leq$$

$$h^{-1}\sup_j\Big(\frac{|\Delta_j D^{(1)}|}{\sqrt{hlog\frac{1}{h}}}\Big)^2 hlog\frac{1}{h}\sum_{j=1}^{n}(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{\Delta_j N^{(1)}\neq 0\}}1_{\{\Delta_j N^{(2)}\neq 0\}}1_{\{|\Delta_j \tilde{J}_2^{(1)}|\leq 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}\leq$$

$$K_1^2(\omega)log\frac{1}{h}\sum_{j=1}^{n}(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{\Delta_j N^{(1)}\neq 0\}}1_{\{|\Delta_j \tilde{J}_2^{(1)}|\leq 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}.$$

The second factor approaches to zero in $\mathcal{L}^1$. In fact

$$E|log\frac{1}{h}\sum_{j=1}^{n}(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{\Delta_j N^{(1)}\neq 0\}}1_{\{|\Delta_j \tilde{J}_2^{(1)}|\leq 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}|\leq$$

$$log\frac{1}{h}\sum_{j=1}^{n}E|(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{\Delta_j N^{(1)}\neq 0\}}1_{\{|\Delta_j \tilde{J}_2^{(1)}|\leq 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}|\leq$$

$$log\frac{1}{h}\sum_{j=1}^{n}E1_{\{\Delta_1 N^{(1)}\neq 0\}}E[(\Delta_1 \tilde{J}_2^{(2)})^2 1_{\{|\Delta_1 \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}]=$$

$$log\frac{1}{h}nP(\Delta_1 N^{(1)}\neq 0)E[(\Delta_1 \tilde{J}_2^{(2)})^2 1_{\{|\Delta_1 \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}]\leq$$

$$T\lambda_1 log\frac{1}{h}E(\Delta_1 \tilde{J}_2^{(2)})^2 = T\lambda_1 hlog\frac{1}{h}\eta_2^2(1)\to 0.$$

For the first term of (4.11) we can write

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\leq 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}\leq$$

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}\leq \sup_j\frac{(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}}{h}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2.$$

Since $\sum_{j=1}^n(\Delta_j D^{(1)})^2=\sum_{j=1}^n(D^{(1)}_{t_j}-D^{(1)}_{t_{j-1}})^2\xrightarrow{P}\int_0^T(\sigma_t^{(1)})^2dt$ which is finite by hypothesis, it suffices to prove that the first factor converges to zero. We have

$$E|\sup_j\frac{(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}}{h}|=E\sup_j\frac{(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}}{h}\leq$$

$$E\sup_j\frac{(\int_{t_{j-1}}^{t_j}\int_{|x|\leq 2\sqrt{r_h}}x\tilde{\mu}^{(2)}(dt,dx)-\int_{t_{j-1}}^{t_j}\int_{2\sqrt{r_h}<|x|\leq 1}xd\nu^{(2)}(dx)dt)^2}{h}\leq$$

$$2E\sup_j \frac{(\int_{t_{j-1}}^{t_j}\int_{|x|\le 2\sqrt{r_h}} x\tilde{\mu}^{(2)}(dt,dx))^2}{h} + 2E\sup_j \frac{(\int_{t_{j-1}}^{t_j}\int_{2\sqrt{r_h}<|x|\le 1} x d\nu^{(2)}(dx)dt)^2}{h} =$$

$$2\sup_j \frac{E(\Delta_j \tilde{J}_{2m}^{(2)})^2}{h} + 2\sup_j \frac{E(\Delta_j \tilde{J}_{2c}^{(2)})^2}{h} = 2\frac{h\eta_2^2(2\sqrt{r_h})}{h} + 2\frac{h^2(c - cr_h^{\frac{1-\alpha_2}{2}})^2}{h} \to 0$$

by using remark 4.3.2 and observing that if we choose $r_h = h^\beta$ with $\beta \in]0,1[$, as usually happens, we have

$$h(c - ch^{\frac{\beta(1-\alpha_2)}{2}})^2 = c^2h - 2c^2h^{1+\frac{\beta(1-\alpha_2)}{2}} + c^2h^{1+\beta(1-\alpha_2)},$$

which tends to zero if $1 + \beta(1-\alpha_2) > 0$. Now, if $\alpha_2 < 1$ this is immediately true; otherwise, if $\alpha_2 \ge 1$ it must be $\beta < \frac{1}{\alpha_2 - 1}$. But $\frac{1}{\alpha_2-1} \in]1,\infty[$, as required.

To show that the other terms of (4.9) tend to zero, taking into account the proof of theorem 4.3.3 and what done up till now, we only need to prove the convergence to zero of three quantities for each term. Consider the third one

$$|h^{-1}\sum_{j=1}^n 2(\Delta_j Y^{(1)})^2(\Delta_j Y^{(2)})(\Delta_j \tilde{J}_2^{(2)}) 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h\}}|.$$

It tends to zero if

$$|h^{-1}\sum_{j=1}^n 2(\Delta_j Y^{(1)})^2(\Delta_j Y^{(2)})(\Delta_j \tilde{J}_2^{(2)}) 1_{\{|\Delta_j X^{(1)}|\le\sqrt{r_h}, |\Delta_j \tilde{J}_2^{(1)}|>2\sqrt{r_h}\}} 1_{\{|\Delta_j X^{(2)}|\le\sqrt{r_h}, |\Delta_j \tilde{J}_2^{(2)}|>2\sqrt{r_h}\}}| \xrightarrow{P} 0,$$

$$h^{-1}\sum_{j=1}^n 2(\Delta_j D^{(1)})^2(\Delta_j D^{(2)})(\Delta_j \tilde{J}_2^{(2)}) 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}} \xrightarrow{P} 0,$$

We know that the first one tends to zero for the presence of the event $|\Delta_j \tilde{J}_2^{(q)}| > 2\sqrt{r_h}$, $q = 1,2$, whereas for the second one we have

$$h^{-1}\sum_{j=1}^n (\Delta_j D^{(1)})^2(\Delta_j D^{(2)})(\Delta_j \tilde{J}_2^{(2)}) 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}} \le$$

$$\sqrt{h^{-1}\sum_{j=1}^n (\Delta_j D^{(2)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}}}\sqrt{h^{-1}\sum_{j=1}^n (\Delta_j D^{(1)})^4},$$

where the first factor tends to zero in probability, while the second one is finite $P - a.s.$ In fact, $\sum_{j=1}^n (\Delta_j D^{(1)})^2$ has a finite limit in probability, then every power variation process of order greater than 2 tends to zero in probability unless is multiply by an appropriate power of h. In this case, $h^{-1}\sum_{j=1}^n (\Delta_j D^{(1)})^4 \xrightarrow{P} [D^{(1)}]_T^{[4]} = \int_0^T (\sigma_t^{(1)})^4 dt$ which is finite by hypothesis. We deal with the fifth term of (4.9). We can write

$$|h^{-1}\sum_{j=1}^n (\Delta_j \tilde{J}_2^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h\}}| \le$$

$$|h^{-1}\sum_{j=1}^n (\Delta_j \tilde{J}_2^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}}| +$$

$$|h^{-1}\sum_{j=1}^n (\Delta_j \tilde{J}_2^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 4r_h\}}| +$$

$$|h^{-1}\sum_{j=1}^n (\Delta_j \tilde{J}_2^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 \le 4r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h, (\Delta_j \tilde{J}_2^{(1)})^2 > 4r_h\}}| +$$

$$|h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_2^{(1)})^2(\Delta_j\tilde{J}_2^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2\le r_h,(\Delta_j\tilde{J}_2^{(1)})^2>4r_h\}}1_{\{(\Delta_j X^{(2)})^2\le r_h,(\Delta_j\tilde{J}_2^{(1)})^2\le 4r_h\}}|.$$

The last three terms tend to zero in probability as always, while for the first one we have

$$|h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_2^{(1)})^2(\Delta_j\tilde{J}_2^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2\le r_h,(\Delta_j\tilde{J}_2^{(1)})^2\le 4r_h\}}1_{\{(\Delta_j X^{(2)})^2\le r_h,(\Delta_j\tilde{J}_2^{(1)})^2\le 4r_h\}}| \le$$

$$|h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_2^{(1)})^2 1_{\{|\Delta_j\tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}(\Delta_j\tilde{J}_2^{(2)})^2 1_{\{|\Delta_j\tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}| =$$

$$h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2m}^{(1)}-\Delta_j\tilde{J}_{2c}^{(1)})^2(\Delta_j\tilde{J}_{2m}^{(2)}-\Delta_j\tilde{J}_{2c}^{(2)})^2 \le$$

$$2h^{-1}\sum_{j=1}^{n}[(\Delta_j\tilde{J}_{2m}^{(1)})^2+(\Delta_j\tilde{J}_{2c}^{(1)})^2][(\Delta_j\tilde{J}_{2m}^{(2)})^2+(\Delta_j\tilde{J}_{2c}^{(2)})^2] =$$

$$2h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2m}^{(1)})^2(\Delta_j\tilde{J}_{2m}^{(2)})^2+2h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2m}^{(1)})^2(\Delta_j\tilde{J}_{2c}^{(2)})^2+$$

$$2h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2c}^{(1)})^2(\Delta_j\tilde{J}_{2m}^{(2)})^2+2h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2c}^{(1)})^2(\Delta_j\tilde{J}_{2c}^{(2)})^2.$$

Each term tends to zero in probability. In fact,

$$h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2m}^{(1)})^2(\Delta_j\tilde{J}_{2m}^{(2)})^2 \le \sup_j\frac{(\Delta_j\tilde{J}_{2m}^{(1)})^2}{h}\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2m}^{(2)})^2.$$

Now

$$E\sup_j\frac{(\Delta_j\tilde{J}_{2m}^{(1)})^2}{h}=\sup_j\frac{E(\Delta_j\tilde{J}_{2m}^{(1)})^2}{h}=\frac{h\eta_2^2(2\sqrt{r_h})}{h}\to 0.$$

and

$$E\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2m}^{(2)})^2=nE(\Delta_1\tilde{J}_{2m}^{(2)})^2=nh\eta_2^2(2\sqrt{r_h})=T\eta_2^2(2\sqrt{r_h})\to 0.$$

Moreover

$$h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2c}^{(1)})^2(\Delta_j\tilde{J}_{2m}^{(2)})^2 \le \sup_j\frac{(\Delta_j\tilde{J}_{2m}^{(2)})^2}{h}\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2c}^{(1)})^2.$$

The second factor is deterministic and tends to zero because

$$\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2c}^{(1)})^2=n(\Delta_1\tilde{J}_{2c}^{(2)})^2=nh^2(c-cr_h^{\frac{1-\alpha_2}{2}})^2=Th(c-cr_h^{\frac{1-\alpha_2}{2}})^2\to 0.$$

Finally

$$h^{-1}\sum_{j=1}^{n}(\Delta_j\tilde{J}_{2c}^{(1)})^2(\Delta_j\tilde{J}_{2c}^{(2)})^2=h^{-1}n(\Delta_1\tilde{J}_{2c}^{(1)})^2(\Delta_1\tilde{J}_{2c}^{(2)})^2$$

$$=h^{-1}nh^2(c-cr_h^{\frac{1-\alpha_1}{2}})^2h^2(c-cr_h^{\frac{1-\alpha_2}{2}})^2=h(c-cr_h^{\frac{1-\alpha_1}{2}})^2h(c-cr_h^{\frac{1-\alpha_2}{2}})^2\to 0,$$

because if $\alpha_1>1$ and $\alpha_2>1$ then $2+\beta(2-\alpha_1-\alpha_2)>0$ if $\beta<\frac{2}{\alpha_1+\alpha_2-2}\in]1,\infty[$, as required.

Now, we deal with the sixth term of (4.9). We have to prove that

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2(\Delta_j\tilde{J}_2^{(1)})^2(\Delta_j\tilde{J}_2^{(2)})1_{\{\Delta_j N^{(1)}\ne 0,|\Delta_j\tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}1_{\{\Delta_j N^{(2)}\ne 0,|\Delta_j\tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}}\xrightarrow{P} 0,$$

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2(\Delta_j \tilde{J}_2^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})1_{\{|\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}} \xrightarrow{P} 0.$$

We have

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2(\Delta_j \tilde{J}_2^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})1_{\{\Delta_j N^{(1)}\neq 0, |\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}1_{\{\Delta_j N^{(2)}\neq 0, |\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}} \le$$

$$h^{-1}K_1^2(\omega)hlog\frac{1}{h}\sum_{j=1}^{n}(\Delta_j \tilde{J}_2^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})1_{\{\Delta_j N^{(1)}\neq 0, |\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}1_{\{\Delta_j N^{(2)}\neq 0, |\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}} \le$$

$$K_1^2(\omega)log\frac{1}{h}\sup_j[(\Delta_j \tilde{J}_2^{(1)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}]\sum_{j=1}^{n}(\Delta_j \tilde{J}_2^{(2)})1_{\{\Delta_j N^{(1)}\neq 0\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}}.$$

The second factor converges to zero in $\mathcal{L}^1$ as always; besides recalling that $\sup_j \frac{(\Delta_j \tilde{J}_2^{(1)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}}{h} \xrightarrow{P} 0$ we conclude that $log\frac{1}{h}\sup_j[(\Delta_j \tilde{J}_2^{(1)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}] \xrightarrow{P} 0$. Moreover

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2(\Delta_j \tilde{J}_2^{(1)})^2(\Delta_j \tilde{J}_2^{(2)})1_{\{|\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}} \le$$

$$\sup_j |\Delta_j \tilde{J}_2^{(2)}|1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}}h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})^2(\Delta_j \tilde{J}_2^{(1)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}} \to 0.$$

As concerning with eighth term of (4.9), we have to show that

$$h^{-1}\sum_{j=1}^{n}2(\Delta_j D^{(1)})(\Delta_j \tilde{J}_2^{(1)})(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{\Delta_j N^{(1)}\neq 0, |\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}1_{\{\Delta_j N^{(2)}\neq 0, |\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}} \xrightarrow{P} 0,$$

and

$$h^{-1}\sum_{j=1}^{n}2(\Delta_j D^{(1)})(\Delta_j \tilde{J}_2^{(1)})(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}} \xrightarrow{P} 0.$$

For the first one we have

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})(\Delta_j \tilde{J}_2^{(1)})(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{\Delta_j N^{(1)}\neq 0, |\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}1_{\{\Delta_j N^{(2)}\neq 0, |\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}} \le$$

$$K_1(\omega)\frac{\sqrt{hlog\frac{1}{h}}}{h}h^2\left(\prod_{q=1}^{2}\left[\frac{\sup_j |\Delta_j \tilde{J}_2^{(q)}|1_{\{|\Delta_j \tilde{J}_2^{(q)}|\le 2\sqrt{r_h}\}}}{h}\right]\right)N_T^{(1)} \to 0.$$

As for the last term of (4.9). We see that

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})(\Delta_j D^{(2)})(\Delta_j \tilde{J}_2^{(1)})(\Delta_j \tilde{J}_2^{(2)})1_{\{\Delta_j N^{(1)}\neq 0, |\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}1_{\{\Delta_j N^{(2)}\neq 0, |\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}} \xrightarrow{P} 0.$$

In fact,

$$h^{-1}\sum_{j=1}^{n}(\Delta_j D^{(1)})(\Delta_j D^{(2)})(\Delta_j \tilde{J}_2^{(1)})(\Delta_j \tilde{J}_2^{(2)})1_{\{\Delta_j N^{(1)}\neq 0, |\Delta_j \tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}1_{\{\Delta_j N^{(2)}\neq 0, |\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}} \le$$

$$K_1(\omega)K_2(\omega)h^2 log\frac{1}{h}\left(\prod_{q=1}^{2}\left[\frac{\sup_j |\Delta_j \tilde{J}_2^{(q)}|1_{\{|\Delta_j \tilde{J}_2^{(q)}|\le 2\sqrt{r_h}\}}}{h}\right]\right)N_T^{(1)} \to 0.$$

This concludes the proof of the first statement of the theorem.

Now, we show that $\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T \xrightarrow{P} \int_0^T \rho^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt$. We have

$$|\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T \xrightarrow{P} \int_0^T \rho^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt| \leq$$

$$\left|h^{-1}\sum_{j=1}^{n-1}[\prod_{q=1}^{2}\Delta_j X^{(q)}1_{\{(\Delta_j X^{(q)})^2\leq r_h\}}\prod_{q=1}^{2}\Delta_{j+1}X^{(q)}1_{\{(\Delta_{j+1}X^{(q)})^2\leq r_h\}}] - \int_0^T \rho^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt\right| =$$

$$|h^{-1}\sum_{j=1}^{n-1}[(\Delta_j Y^{(1)}+\Delta_j\tilde{J}_2^{(1)})1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}(\Delta_{j+1}Y^{(1)}+\Delta_{j+1}\tilde{J}_2^{(1)})1_{\{(\Delta_{j+1}X^{(q)})^2\leq r_h\}}\times$$
$$\times(\Delta_j Y^{(2)}+\Delta_j\tilde{J}_2^{(2)})1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}(\Delta_{j+1}Y^{(2)}+\Delta_{j+1}\tilde{J}_2^{(2)})1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}]| \leq$$

$$|h^{-1}\sum_{j=1}^{n-1}[\prod_{q=1}^{2}\Delta_j Y^{(q)}1_{\{(\Delta_j X^{(q)})^2\leq r_h\}}\prod_{q=1}^{2}\Delta_{j+1}Y^{(q)}1_{\{(\Delta_{j+1}X^{(q)})^2\leq r_h\}}] - \int_0^T \rho^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j Y^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}Y^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j Y^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_j\tilde{J}_2^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j Y^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}Y^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j\tilde{J}_2^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}Y^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j Y^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}Y^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j\tilde{J}_2^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j Y^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j Y^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}Y^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j Y^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j Y^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j Y^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j\tilde{J}_2^{(1)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}Y^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j Y^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j\tilde{J}_2^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j\tilde{J}_2^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}Y^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j Y^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}Y^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j\tilde{J}_2^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}Y^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j Y^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j\tilde{J}_2^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}Y^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j\tilde{J}_2^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}Y^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j\tilde{J}_2^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}Y^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j\tilde{J}_2^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j\tilde{J}_2^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j Y^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}Y^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j\tilde{J}_2^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j Y^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j\tilde{J}_2^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j\tilde{J}_2^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}Y^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_j\tilde{J}_2^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\leq r_h\}}\Delta_j\tilde{J}_2^{(2)}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}}\Delta_{j+1}\tilde{J}_2^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\leq r_h\}}|. \tag{4.12}$$

Now, adding and subtracting

$$h^{-1}\sum_{j=1}^{n-1}[\prod_{q=1}^{2}\Delta_jY^{(q)}1_{\{(\Delta_jY^{(q)})^2\le 4r_h\}}\prod_{q=1}^{2}\Delta_{j+1}Y^{(q)}1_{\{(\Delta_{j+1}Y^{(q)})^2\le 4r_h\}}],$$

to the first term of the right hand side, we obtain

$$|h^{-1}\sum_{j=1}^{n-1}[\prod_{q=1}^{2}\Delta_jY^{(q)}1_{\{(\Delta_jX^{(q)})^2\le r_h\}}\prod_{q=1}^{2}\Delta_{j+1}Y^{(q)}1_{\{(\Delta_{j+1}X^{(q)})^2\le r_h\}}]-\int_0^T\rho^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt|\le$$

$$|h^{-1}\sum_{j=1}^{n-1}[\prod_{q=1}^{2}\Delta_jY^{(q)}1_{\{(\Delta_jY^{(q)})^2\le 4r_h\}}\prod_{q=1}^{2}\Delta_{j+1}Y^{(q)}1_{\{(\Delta_{j+1}Y^{(q)})^2\le 4r_h\}}]-\int_0^T\rho^2(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt|+$$

$$|h^{-1}\sum_{j=1}^{n-1}\Delta_jY^{(1)}\Delta_{j+1}Y^{(1)}\Delta_jY^{(1)}\Delta_j\tilde{J}_2^{(2)}\times$$

$$\times(1_{\{(\Delta_jX^{(1)})^2\le r_h,(\Delta_{j+1}X^{(1)})^2\le r_h,(\Delta_jX^{(2)})^2\le r_h,(\Delta_{j+1}X^{(2)})^2\le r_h\}}+$$

$$-1_{\{(\Delta_jY^{(1)})^2\le 4r_h,(\Delta_{j+1}Y^{(1)})^2\le 4r_h,(\Delta_jY^{(2)})^2\le 4r_h,(\Delta_{j+1}Y^{(2)})^2\le 4r_h\}})|$$

The first term tends to zero in probability by theorem 4.2.1, whereas for the second one and all others in (4.12), according to what we have done until now, we only show the convergence to zero for some of them because they have a similar behaviuor for small h. Really, in this proof, we repeatedly use the Cauchy-Schwartz inequality in such a way that to exploit the previous proof taking into account that

$$|\sum_{j=1}^{n-1}x_j|\le\sum_{j=1}^{n-1}|x_j|\le\sum_{j=1}^{n}|x_j|$$

We only study terms whose convergence to zero is not immediate. They are

$$|h^{-1}\sum_{j=1}^{n-1}[\prod_{q=1}^{2}\Delta_jY^{(q)}1_{\{(\Delta_jX^{(q)})^2\le r_h,\Delta_jY^{(q)})^2\ge 4r_h\}}]\times$$

$$\times[\prod_{q=1}^{2}\Delta_{j+1}Y^{(q)}1_{\{(\Delta_{j+1}X^{(q)})^2\le r_h,\Delta_{j+1}Y^{(q)})^2\ge 4r_h\}}]|, \tag{4.13}$$

$$|h^{-1}\sum_{j=1}^{n-1}[\prod_{q=1}^{2}\Delta_jY^{(q)}1_{\{(\Delta_jX^{(q)})^2> r_h,\Delta_jY^{(q)})^2\le 4r_h\}}]\times$$

$$\times[\prod_{q=1}^{2}\Delta_{j+1}Y^{(q)}1_{\{(\Delta_{j+1}X^{(q)})^2> r_h,\Delta_{j+1}Y^{(q)})^2\le 4r_h\}}]|, \tag{4.14}$$

$$|h^{-1}\sum_{j=1}^{n-1}[\Delta_jY^{(1)}1_{\{(\Delta_jX^{(1)})^2\le r_h\}}\Delta_{j+1}Y^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\le r_h\}}\times$$

$$\times\Delta_jY^{(2)}1_{\{(\Delta_jX^{(2)})^2\le r_h\}}\Delta_{j+1}\tilde{J}_2^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\le r_h\}}]| \tag{4.15}$$

Let's begin by (4.13). Since

$$|h^{-1}\sum_{j=1}^{n-1}\prod_{q=1}^{2}\Delta_jY^{(q)}1_{\{(\Delta_jX^{(q)})^2\le r_h,\Delta_jY^{(q)})^2\ge 4r_h\}}\prod_{q=1}^{2}\Delta_{j+1}Y^{(q)}1_{\{(\Delta_{j+1}X^{(q)})^2\le r_h,\Delta_{j+1}Y^{(q)})^2\ge 4r_h\}}|\le$$

$$|\sqrt{h^{-1}\sum_{j=1}^{n-1}\prod_{q=1}^{2}(\Delta_jY^{(q)})^21_{\{(\Delta_jX^{(q)})^2\le r_h,\Delta_jY^{(q)})^2\ge 4r_h\}}}\times$$

$$\sqrt{h^{-1}\sum_{j=1}^{n-1}\prod_{q=1}^{2}(\Delta_{j+1}Y^{(q)})^2 1_{\{(\Delta_{j+1}X^{(q)})^2\le r_h,\Delta_{j+1}Y^{(q)})^2\ge 4r_h\}}}|.$$

The convergence to zero is immediate by the first part of the theorem. The same holds for (4.14) if we remark that

$$|h^{-1}\sum_{j=1}^{n-1}[\prod_{q=1}^{2}\Delta_j Y^{(q)}1_{\{(\Delta_j X^{(q)})^2>r_h,\Delta_j Y^{(q)})^2\le 4r_h\}}][\prod_{q=1}^{2}\Delta_{j+1}Y^{(q)}1_{\{(\Delta_{j+1}X^{(q)})^2>r_h,\Delta_{j+1}Y^{(q)})^2\le 4r_h\}}]|\le$$

$$|\sqrt{h^{-1}\sum_{j=1}^{n-1}\prod_{q=1}^{2}(\Delta_j Y^{(q)})^2 1_{\{(\Delta_j X^{(q)})^2>r_h,\Delta_j Y^{(q)})^2\le 4r_h\}}}\times$$

$$\times\sqrt{h^{-1}\sum_{j=1}^{n-1}\prod_{q=1}^{2}(\Delta_{j+1}Y^{(q)})^2 1_{\{(\Delta_{j+1}X^{(q)})^2>r_h,\Delta_{j+1}Y^{(q)})^2\le 4r_h\}}}|.$$

We can still proceed at the same way for (4.15); in fact

$$|h^{-1}\sum_{j=1}^{n-1}[\Delta_j Y^{(1)}1_{\{(\Delta_j X^{(1)})^2\le r_h\}}\Delta_{j+1}Y^{(1)}1_{\{(\Delta_{j+1}X^{(1)})^2\le r_h\}}\times$$

$$\times\Delta_j Y^{(2)}1_{\{(\Delta_j X^{(2)})^2\le r_h\}}\Delta_{j+1}\tilde{J}_2^{(2)}1_{\{(\Delta_{j+1}X^{(2)})^2\le r_h\}}]|\le$$

$$|\sqrt{h^{-1}\sum_{j=1}^{n-1}(\Delta_{j+1}Y^{(1)})^2(\Delta_{j+1}\tilde{J}_2^{(2)})^2 1_{\{(\Delta_{j+1}X^{(1)})^2\le r_h\}}1_{\{(\Delta_{j+1}X^{(2)})^2\le r_h\}}}\times$$

$$\times\sqrt{h^{-1}\sum_{j=1}^{n-1}(\Delta_j Y^{(1)})^2(\Delta_j Y^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2\le r_h\}}1_{\{(\Delta_j X^{(2)})^2\le r_h\}}}|,$$

approaches zero since

$$h^{-1}\sum_{j=1}^{n-1}(\Delta_{j+1}Y^{(1)})^2(\Delta_{j+1}\tilde{J}_2^{(2)})^2 1_{\{(\Delta_{j+1}X^{(1)})^2\le r_h\}}1_{\{(\Delta_{j+1}X^{(2)})^2\le r_h\}}\xrightarrow{P}0,$$

as in the first part of the proof, whereas

$$h^{-1}\sum_{j=1}^{n-1}(\Delta_j Y^{(1)})^2(\Delta_j Y^{(2)})^2 1_{\{(\Delta_j X^{(1)})^2\le r_h\}}1_{\{(\Delta_j X^{(2)})^2\le r_h\}}\sim$$

$$\sim h^{-1}\sum_{j=1}^{n-1}(\Delta_j D^{(1)})^2(\Delta_j D^{(2)})^2 1_{\{|\Delta_j\tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}}1_{\{|\Delta_j\tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}}\le$$

$$h^{-1}\sum_{j=1}^{n-1}(\Delta_j D^{(1)})^2(\Delta_j D^{(2)})^2=v_{2,2}^{(n)}(D^{(1)},D^{(2)})_T\xrightarrow{P}\int_0^T(2\rho^2+1)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt<\infty,$$

by assumption. This concludes the proof since the other terms would be treated at the same way.

•

4.3.2 Work in progress: the Central Limit Theorem

Differently from the case of finite activity jump component, a Central Limit Theorem for a normalized version of the threshold estimator only holds for a restricted class of Lévy processes and precisely for those one which are α-stable. To define it, recall that if X is a random variable taking values in $\mathbb{R}$ with distribution function P_X its characteristic function (Fourier transform) is given by

$$\widehat{P}_X(z) = \int_{\mathbb{R}} e^{izx} \widehat{P}_X(dx), \quad z \in \mathbb{R}.$$

Definition 4.3.5. *A random variable X is said to have a* stable distribution *if $\forall a > 0$, $\exists b(a) > 0$ and $c(a) \in \mathbb{R}$ such that*

$$\widehat{P}_X(z)^a = \widehat{P}_X(zb(a))e^{izc(a)}, \quad \forall z \in \mathbb{R},$$

whereas it has a strictly stable distribution *if*

$$\widehat{P}_X(z)^a = \widehat{P}_X(zb(a)), \quad \forall z \in \mathbb{R}$$

Definition 4.3.6. *A Lévy process L is said to be strictly α-stable, for $\alpha \in]0,2]$, if it satisfies*

$$(\frac{L_{ct}}{c^{1/\alpha}})_{t\geq 0} \stackrel{D}{=} (L_t)_{t\geq 0}, \quad \forall c > 0.$$

Proposition 4.3.7. *A distribution on $\mathbb{R}^d$ is α-stable with $\alpha \in]0,2[$ if and only if it is infinitely divisible and there exists a finite measure λ on the unit sphere S of $\mathbb{R}^d$ such that*

$$\nu(A) = \int_S \lambda(dy) \int_{\mathbb{R}_+} \frac{1_A(ry)}{r^{1+\alpha}} dr.$$

Theorem 4.3.8. *(Central Limit Theorem) If the conditions of theorem 4.3.3 are held, if $\sigma_t^{(q)} > 0$ $\forall t \in [0,T]$, $q = 1,2$, and if the jump components of $X^{(1)}$ and $X^{(2)}$ are both α-stable processes with $\alpha_1 = \alpha_2 = 0$, then*

$$\frac{h^{-1/2}(\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\tilde{v}_{2,2}^{(n)}(X^{(1)},X^{(2)})_T - \tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}} \xrightarrow{D} N(0,1).$$

***Proof*.** Firstly, we remark that

$$\mathrm{D}lim_h \frac{h^{-1/2}(\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\tilde{v}_{2,2}^{(n)}(X^{(1)},X^{(2)})_T - \tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}} = \mathrm{D}lim_h \frac{h^{-1/2}(\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt}}$$

Moreover

$$\mathrm{D}lim_h \frac{h^{-1/2}(\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt}} =$$

$$\mathrm{D}lim_h \frac{h^{-1/2}(\sum_{j=1}^n \Delta_j X^{(1)} 1_{\{(\Delta_j X^{(1)})^2 \leq r_h\}} \Delta_j X^{(2)} 1_{\{(\Delta_j X^{(2)})^2 \leq r_h\}} - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt}} =$$

$$\mathrm{D}lim_h \frac{h^{-1/2}\sum_{j=1}^n \Delta_j D^{(1)} \Delta_j D^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \leq r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \leq r_h\}} - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt}} +$$

$$\mathrm{D}lim_h \frac{h^{-1/2}\sum_{j=1}^n \Delta_j D^{(1)} \Delta_j J^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \leq r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \leq r_h\}}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2 dt}} +$$

$$\mathrm{D}lim_h \frac{h^{-1/2} \sum_{j=1}^n \Delta_j J^{(1)} \Delta_j D^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h\}}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt}} +$$

$$\mathrm{D}lim_h \frac{h^{-1/2} \sum_{j=1}^n \Delta_j J^{(1)} \Delta_j J^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h\}}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt}}$$

Since $1_{\{(\Delta_j X^{(q)})^2 \le r_h\}} = 1 - 1_{\{(\Delta_j X^{(q)})^2 > r_h\}}$ we get

$$\mathrm{D}lim_h \frac{h^{-1/2} (\tilde{v}_{1,1}^{(n)} (X^{(1)}, X^{(2)})_T - \int_0^T \rho \sigma_t^{(1)} \sigma_t^{(2)} dt)}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt}} =$$

$$\mathrm{D}lim_h \frac{h^{-1/2} (\sum_{j=1}^n \Delta_j D^{(1)} \Delta_j D^{(2)} - \int_0^T \rho \sigma_t^{(1)} \sigma_t^{(2)} dt)}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt}} +$$

$$\mathrm{D}lim_h \frac{h^{-1/2} \sum_{j=1}^n \Delta_j D^{(1)} \Delta_j D^{(2)} 1_{\{(\Delta_j X^{(1)})^2 > r_h\}} 1_{\{(\Delta_j X^{(2)})^2 > r_h\}}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt}} +$$

$$\mathrm{D}lim_h \frac{h^{-1/2} \sum_{j=1}^n \Delta_j D^{(1)} \Delta_j D^{(2)} 1_{\{(\Delta_j X^{(1)})^2 > r_h\}}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt}} +$$

$$\mathrm{D}lim_h \frac{h^{-1/2} \sum_{j=1}^n \Delta_j D^{(1)} \Delta_j D^{(2)} 1_{\{(\Delta_j X^{(2)})^2 > r_h\}}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt}} +$$

$$\mathrm{D}lim_h \frac{h^{-1/2} \sum_{j=1}^n \Delta_j D^{(1)} \Delta_j J^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h\}}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt}} +$$

$$\mathrm{D}lim_h \frac{h^{-1/2} \sum_{j=1}^n \Delta_j J^{(1)} \Delta_j D^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h\}}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt}} +$$

$$\mathrm{D}lim_h \frac{h^{-1/2} \sum_{j=1}^n \Delta_j J^{(1)} \Delta_j J^{(2)} 1_{\{(\Delta_j X^{(1)})^2 \le r_h\}} 1_{\{(\Delta_j X^{(2)})^2 \le r_h\}}}{\sqrt{\int_0^T (1+\rho^2)(\sigma_t^{(1)})^2 (\sigma_t^{(2)})^2 dt}} \tag{4.16}$$

The first term of the right hand side of (4.16) tends in distribution to a standard Normal law by Proposition 3.3.5, while the other terms tend to zero in probability and so in distribution. For each one we can neglect the denominator since it is a finite quantity. The second, the third and the fourth terms tend to zero in the same way. Consider the second one. We have to prove that

$$h^{-1/2} \sum_{j=1}^n \Delta_j D^{(1)} \Delta_j D^{(2)} 1_{\{(\Delta_j X^{(1)})^2 > r_h\}} 1_{\{(\Delta_j X^{(2)})^2 > r_h\}} \xrightarrow{P} 0.$$

We have

$$h^{-1/2} \sum_{j=1}^n \Delta_j D^{(1)} \Delta_j D^{(2)} 1_{\{(\Delta_j X^{(1)})^2 > r_h\}} 1_{\{(\Delta_j X^{(2)})^2 > r_h\}} \le$$

$$h^{-1/2} \sum_{j=1}^n \Delta_j D^{(1)} \Delta_j D^{(2)} (1_{\{|\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}\}} + 1_{\{\Delta_j N^{(1)} \ne 0\}})(1_{\{|\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}\}} + 1_{\{\Delta_j N^{(1)} \ne 0\}}) =$$

$$h^{-1/2} \sum_{j=1}^n \Delta_j D^{(1)} \Delta_j D^{(2)} 1_{\{|\Delta_j \tilde{J}_2^{(1)}| > 2\sqrt{r_h}\}} 1_{\{|\Delta_j \tilde{J}_2^{(2)}| > 2\sqrt{r_h}\}} +$$

$$h^{-1/2}\sum_{j=1}^{n}\Delta_j D^{(1)}\Delta_j D^{(2)}1_{\{\Delta_j N^{(1)}\neq 0\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|>2\sqrt{r_h}\}}+$$

$$h^{-1/2}\sum_{j=1}^{n}\Delta_j D^{(1)}\Delta_j D^{(2)}1_{\{|\Delta_j \tilde{J}_2^{(1)}|>2\sqrt{r_h}\}}1_{\{\Delta_j N^{(2)}\neq 0\}}+$$

$$h^{-1/2}\sum_{j=1}^{n}\Delta_j D^{(1)}\Delta_j D^{(2)}1_{\{\Delta_j N^{(1)}\neq 0\}}1_{\{\Delta_j N^{(2)}\neq 0\}},$$

where for the first inequality we use lemma 3.5 in (Cont-Mancini, 2005), because $\sup_j|\Delta_j D^{(1)}\Delta_j D^{(2)}| = O(hlog\frac{1}{h})$. The last three terms approach zero. In fact, each one is dominated for some $q = 1, 2$ by

$$K_1(\omega)K_2(\omega)\sqrt{h}log\frac{1}{h}\sum_{j=1}^{n}1_{\{\Delta_j N^{(q)}\neq 0\}} \leq K_1(\omega)K_2(\omega)\sqrt{h}log\frac{1}{h}N_T^{(q)} \to 0, \quad P-a.s.$$

It remains to consider $h^{-1/2}\sum_{j=1}^{n}\Delta_j D^{(1)}\Delta_j D^{(2)}1_{\{|\Delta_j \tilde{J}_2^{(1)}|>2\sqrt{r_h}\}}1_{\{|\Delta_j \tilde{J}_2^{(2)}|>2\sqrt{r_h}\}}$. In order to show that it tends to zero in probability we can prove that a subsequence tends to zero $P-a.s.$. By Cont-Mancini (2006), for small h, for such a subsequence $1_{\{|\Delta_{j_m}\tilde{J}_2^{(q)}|>2\sqrt{r_h}\}} = 0$, $P-a.s.$, $q = 1, 2$, therefore

$$h^{-1/2}\sum_{j_m=1}^{n}\Delta_{j_m} D^{(1)}\Delta_{j_m} D^{(2)}1_{\{|\Delta_{j_m}\tilde{J}_2^{(1)}|>2\sqrt{r_h}\}}1_{\{|\Delta_{j_m}\tilde{J}_2^{(2)}|>2\sqrt{r_h}\}} = 0, \quad P-a.s.,$$

and hence the term tends to zero in probability.

Moreover, for the fifth term of (4.16), we see that

$$h^{-1/2}\sum_{j=1}^{n}\Delta_j D^{(1)}\Delta_j J^{(2)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}} \xrightarrow{P} 0,$$

because

$$\mathrm{P}lim_h h^{-1/2}\sum_{j=1}^{n}\Delta_j D^{(1)}\Delta_j J^{(2)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}1_{\{(\Delta_j X^{(2)})^2\leq r_h\}} =$$

$$\mathrm{P}lim_h h^{-1/2}\sum_{j=1}^{n}\Delta_j D^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_j J^{(2)}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h},\Delta_j N^{(2)}=0\}},$$

where we still use lemma 3.5 in Cont-Mancini (Cont-Mancini, 2005), since $|\Delta_j J^{(2)}| \leq 2\sqrt{r_h}$ on the set $\{(\Delta_j X^{(2)})^2 \leq r_h\}$. Therefore

$$\mathrm{P}lim_h h^{-1/2}\sum_{j=1}^{n}\Delta_j D^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_j J^{(2)}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h},\Delta_j N^{(2)}=0\}} =$$

$$\mathrm{P}lim_h h^{-1/2}\sum_{j=1}^{n}\Delta_j D^{(1)}1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_j \tilde{J}_2^{(2)}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}$$

$$\mathrm{P}lim_h h^{-1/2}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j} a_s^{(1)}ds + \int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_j \tilde{J}_2^{(2)}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}}$$

Since $\sup_{t\in[0,T]}|a_t^{(1)}|(\omega) < \infty$, we can write

$$\mathrm{P}lim_h h^{-1/2}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j} a_s^{(1)}ds)1_{\{(\Delta_j X^{(1)})^2\leq r_h\}}\Delta_j \tilde{J}_2^{(2)}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\leq 2\sqrt{r_h}\}} \leq$$

$$\text{P}lim_h\sqrt{h^{-1}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j} a_s^{(1)}ds)^2 1_{\{(\Delta_j X^{(1)})^2\le r_h\}}}\sqrt{\sum_{j=1}^{n}(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}}}\le$$

$$\text{P}lim_h\sqrt{h^{-1}nh^2(\sup_{T\in[0,T]}|a_T^{(1)}|(\omega))^2}\sqrt{\sum_{j=1}^{n}(\Delta_j \tilde{J}_2^{(2)})^2 1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}}}=0,$$

since the first factor is bounded and the second one has the same limit in probability of

$$\int_0^T\int_{|x|\le 2\sqrt{r_h}} x^2\nu^{(2)}(dx)dt = T\eta_2^2(2\sqrt{r_h}),$$

which is zero. Now, consider

$$\text{P}lim_h h^{-1/2}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{(\Delta_j X^{(1)})^2\le r_h\}}\Delta_j\tilde{J}_2^{(2)}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}}.$$

We have

$$\text{P}lim_h h^{-1/2}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{(\Delta_j X^{(1)})^2\le r_h\}}\Delta_j\tilde{J}_2^{(2)}1_{\{|\Delta_j \tilde{J}_2^{(2)}|\le 2\sqrt{r_h}\}}=$$

$$\text{P}lim_h[h^{-1/2}(\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{(\Delta_j X^{(1)})^2\le r_h\}}\Delta_j\tilde{J}_{2m}^{(2)}+$$

$$-\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{(\Delta_j X^{(1)})^2\le r_h\}}\Delta_j\tilde{J}_{2c}^{(2)})].$$

For the second term we can write

$$h^{-1/2}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{(\Delta_j X^{(1)})^2\le r_h\}}\Delta_j\tilde{J}_{2c}^{(2)})]=$$

$$h^{-1/2}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})\Delta_j\tilde{J}_{2c}^{(2)}+h^{-1/2}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{(\Delta_j X^{(1)})^2> r_h\}}\Delta_j\tilde{J}_{2c}^{(2)}.$$

Now since $\Delta_j\tilde{J}_{2c}^{(2)}=hO(c-cr_h^{\frac{1-\alpha_2}{2}})$ we can write

$$h^{-1/2}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})\Delta_j\tilde{J}_{2c}^{(2)}=\sqrt{h}O(c-cr_h^{\frac{1-\alpha_2}{2}})\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})=$$

$$\sqrt{h}O(c-cr_h^{\frac{1-\alpha_2}{2}})(\sigma^{(1)}.W^{(1)})_T\to 0,\quad P-a.s.$$

Moreover, we can apply lemma 3.5 in Cont-Mancini (Cont-Mancini, 2005) to obtain

$$h^{-1/2}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{(\Delta_j X^{(1)})^2> r_h\}}\Delta_j\tilde{J}_{2c}^{(2)}=$$

$$h^{-1/2}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})(1_{\{(\Delta_j \tilde{J}_2^{(1)})^2> 4r_h\}}+1_{\{\Delta_j N^{(1)}\neq 0\}})\Delta_j\tilde{J}_{2c}^{(2)},$$

so that both terms tend to zero

$$h^{-1/2}\sum_{j=1}^{n}(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{\Delta_j N^{(1)}\neq 0\}}\Delta_j\tilde{J}_{2c}^{(2)}\le$$

$$N_T^{(1)}K_1(\omega)\sqrt{h}log\frac{1}{h}\sup_j\frac{\Delta_j\tilde{J}_{2c}^{(2)}}{\sqrt{h}} \to 0, \quad P-a.s.$$

while $h^{-1/2}\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{(\Delta_j\tilde{J}_2^{(1)})^2>4r_h\}}\Delta_j\tilde{J}_{2c}^{(2)}=0$, $P-a.s.$, for small h, because $1_{\{(\Delta_j\tilde{J}_2^{(1)})^2>4r_h\}}=0$, $P-a.s.$, for small h. Indeed

$$\text{P}lim_h h^{-1/2}\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{(\Delta_jX^{(1)})^2\le r_h\}}\Delta_j\tilde{J}_{2m}^{(2)}=$$

$$\text{P}lim_h h^{-1/2}\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})\Delta_j\tilde{J}_{2m}^{(2)}+\text{P}lim_h h^{-1/2}\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})1_{\{(\Delta_jX^{(1)})^2>r_h\}}\Delta_j\tilde{J}_{2m}^{(2)}$$

For the first term it is sufficient to show that the square tends to zero in probability, since $X_n^2\xrightarrow{P}0\Rightarrow X_n\xrightarrow{P}0$. However, since the factors are increments of martingales it suffices to prove the convergence of the sum of squares; we have

$$h^{-1}\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})^2(\Delta_j\tilde{J}_{2m}^{(2)})^2\le$$

$$\sup_j\frac{(\Delta_j\tilde{J}_{2m}^{(2)})^2}{h}\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})^2\xrightarrow{P}0,$$

because $\sup_j\frac{(\Delta_j\tilde{J}_{2m}^{(2)})^2}{h}\xrightarrow{P}0$ by Doob's inequality while $\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})^2\xrightarrow{P}\int_0^T(\sigma_t^{(1)})^2dt$, which is bounded $P-a.s.$ by assumption. Indeed, by lemma 3.5 (Cont-Mancini, 2005)

$$\text{P}lim_h h^{-1/2}\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})(\Delta_j\tilde{J}_{2m}^{(2)})1_{\{(\Delta_jX^{(1)})^2>r_h\}}\le$$

$$\text{P}lim_h h^{-1/2}\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})(\Delta_j\tilde{J}_{2m}^{(2)})(1_{\{\Delta_jN^{(1)}\ne0\}}+1_{\{(\Delta_j\tilde{J}_2^{(1)})^2>4r_h\}})$$

Both terms tend to zero; in fact, the first one contains at most $N_T^{(1)}$ terms

$$h^{-1/2}\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})(\Delta_j\tilde{J}_{2m}^{(2)})1_{\{\Delta_jN^{(1)}\ne0\}}\le K_1^{'}(\omega)\sqrt{hlog\frac{1}{h}}\sup_j\frac{|\Delta_j\tilde{J}_{2m}^{(2)}|}{\sqrt{h}}N_T^{(1)}\to0.$$

For the second one, since there exists a subsequence such that $1_{\{(\Delta_{j_m}\tilde{J}_2^{(1)})^2>4r_h\}}=0$, $P-a.s.$, (Cont-Mancini, 2005), we conclude that

$$h^{-1/2}\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})(\Delta_j\tilde{J}_{2m}^{(2)})1_{\{(\Delta_j\tilde{J}_2^{(1)})^2>4r_h\}}=0, \quad P-a.s.,$$

and so

$$\text{P}lim_h h^{-1/2}\sum_{j=1}^n(\int_{t_{j-1}}^{t_j}\sigma_s^{(1)}dW_s^{(1)})(\Delta_j\tilde{J}_{2m}^{(2)})1_{\{(\Delta_j\tilde{J}_2^{(1)})^2>4r_h\}}=0$$

as required.

•

To conclude the proof it remains to show that

$$\text{D}lim_h\frac{h^{-1/2}\sum_{j=1}^n\Delta_jJ^{(1)}\Delta_jJ^{(2)}1_{\{(\Delta_jX^{(1)})^2\le r_h\}}1_{\{(\Delta_jX^{(2)})^2\le r_h\}}}{\sqrt{\int_0^T(1+\rho^2)(\sigma_t^{(1)})^2(\sigma_t^{(2)})^2dt}}=0,$$

which is implied by

$$\mathrm{D}lim_h \frac{h^{-1/2}\sum_{j=1}^{n}\Delta_j\tilde{J}_2^{(1)}1_{\{(\Delta_j\tilde{J}_2^{(1)})^2\le 4r_h\}}\Delta_j\tilde{J}_2^{(2)}1_{\{(\Delta_j\tilde{J}_2^{(2)})^2\le 4r_h\}}}{\sqrt{h}}=0$$

The situation is very complicated and a possible approach requires the application of the following celebrate Central Limit Theorem on the triangular array of r.vs. due to Lindeberg and Feller.

Theorem 4.3.9. *(Lindeberg-Feller) Let $\{H_{nj},\ j=1,....,j_n,\ n=1,2,....\}$ be a double array of r.v.s independent in each row such that $EH_{nj}=0$ and $EH_{nj}^2=\sigma_{nj}^2<\infty$ for each n and j and moreover $\sum_{j=1}^{j_n}\sigma_{nj}^2=1$. Let F_{nj} be the distribution function of H_{nj}. In order that*

1. $\max_{1\le j\le j_n}P(|H_{nj}|>\epsilon)\to 0,\ \forall\epsilon>0,$
2. $\sum_{j=1}^{j_n}H_{nj}\xrightarrow{D}N(0,1),$

it is necessary and sufficient that for each $\eta>0$ the Lindeberg condition holds, i.e.

$$\sum_{j=1}^{j_n}\int_{|x|>\eta}x^2F_{nj}(dx)=\sum_{j=1}^{j_n}EH_{nj}^21_{\{|H_{nj}|>\eta\}}\to 0.$$

Our array is done by the r.vs. $H_{nj}^{'}=\Delta_j\tilde{J}_2^{(1)}1_{\{|\Delta_j\tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}\Delta_j\tilde{J}_2^{(2)}1_{\{\Delta_j\tilde{J}_2^{(2)}\le 2\sqrt{r_h}\}}$. Since the theorem can be applied to centered r.vs. our first objective is to compute expectation of $H_{nj}^{'}$. In particular, we are interested in its speed of convergence to zero. Then

$$E(\Delta_j\tilde{J}_2^{(1)}1_{\{|\Delta_j\tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}\Delta_j\tilde{J}_2^{(2)}1_{\{\Delta_j\tilde{J}_2^{(2)}\le 2\sqrt{r_h}\}})=E[(\Delta_j\tilde{J}_{2m}^{(1)}-\Delta_j\tilde{J}_{2c}^{(1)})(\Delta_j\tilde{J}_{2m}^{(2)}-\Delta_j\tilde{J}_{2c}^{(2)})].$$

The first factor $E(\Delta_j\tilde{J}_{2m}^{(1)}\Delta_j\tilde{J}_{2m}^{(2)})$ is not immediate because it requires the computation of an integral of the type

$$\int_{|x|\le 2\sqrt{r_h},|y|\le 2\sqrt{r_h}}xy\boldsymbol{\nu}(dx,dy),$$

which says to us the speed of convergence of co-jumps of a bi-dimensional pure jumping Lévy process. $\boldsymbol{\nu}$ is the Lévy measure of such a process. The computation of the integral is more simple if we limit ourselves to the case of positive jumps and we assume without loss of generality $r_h=h^\beta$ with $\beta\in]0,1[$. The result is

$$E(\Delta_j\tilde{J}_{2m}^{(1)}\Delta_j\tilde{J}_{2m}^{(2)})=O(h^{\frac{\beta(\alpha_1+\alpha_2-\alpha_1\alpha_2)}{2\alpha_1}}).$$

As regards the other terms, we have $E(\Delta_j\tilde{J}_{2m}^{(1)}\Delta_j\tilde{J}_{2c}^{(2)})=\Delta_j\tilde{J}_{2c}^{(2)}E\Delta_j\tilde{J}_{2m}^{(1)}=0$ and so it does not contribute, whereas

$$\Delta_j\tilde{J}_{2c}^{(1)}\Delta_j\tilde{J}_{2c}^{(2)}=h^2\int_{2\sqrt{r_h}<|x|\le 1}x\nu^{(1)}(dx)\int_{2\sqrt{r_h}<|x|\le 1}x\nu^{(2)}(dx)=$$

$$h^2O(c-ch^{\frac{\beta(1-\alpha_1)}{2}})O(c-ch^{\frac{\beta(1-\alpha_2)}{2}})=O(c^2h^2-c^2h^{2+\frac{\beta(1-\alpha_1)}{2}}-c^2h^{2+\frac{\beta(1-\alpha_2)}{2}}+c^2h^{2+\frac{\beta(2-\alpha_1-\alpha_2)}{2}}),$$

which can be compared with $O(h^{\frac{\beta(\alpha_1+\alpha_2-\alpha_1\alpha_2)}{2\alpha_1}})$. We have two cases

1. $\alpha_1\le\alpha_2<1$. In this case

$$O(c^2h^2-c^2h^{2+\frac{\beta(1-\alpha_1)}{2}}-c^2h^{2+\frac{\beta(1-\alpha_2)}{2}}+c^2h^{2+\frac{\beta(2-\alpha_1-\alpha_2)}{2}})=O(h^2),$$

so that the expectation is

$$E(\Delta_j\tilde{J}_2^{(1)}1_{\{|\Delta_j\tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}\Delta_j\tilde{J}_2^{(2)}1_{\{\Delta_j\tilde{J}_2^{(2)}\le 2\sqrt{r_h}\}})=O(h^{\frac{\beta(\alpha_1+\alpha_2-\alpha_1\alpha_2)}{2\alpha_1}})+O(h^2)=$$

$$=\begin{cases}O(h^{\frac{\beta(\alpha_1+\alpha_2-\alpha_1\alpha_2)}{2\alpha_1}}), & \alpha_1<\frac{\beta\alpha_2}{4-\beta+\beta\alpha_2}\\ O(h^2), & \alpha_1>\frac{\beta\alpha_2}{4-\beta+\beta\alpha_2}\end{cases}$$

2. $1 < \alpha_1 \le \alpha_2$. In this case
$$O(c^2h^2 - c^2h^{2+\frac{\beta(1-\alpha_1)}{2}} - c^2h^{2+\frac{\beta(1-\alpha_2)}{2}} + c^2h^{2+\frac{\beta(2-\alpha_1-\alpha_2)}{2}}) = O(h^{2+\frac{\beta(2-\alpha_1-\alpha_2)}{2}}),$$
so that
$$E(\Delta_j\tilde{J}_2^{(1)}1_{\{|\Delta_j\tilde{J}_2^{(1)}|\le 2\sqrt{r_h}\}}\Delta_j\tilde{J}_2^{(2)}1_{\{\Delta_j\tilde{J}_2^{(2)}\le 2\sqrt{r_h}\}}) =$$
$$O(h^{\frac{\beta(\alpha_1+\alpha_2-\alpha_1\alpha_2)}{2\alpha_1}}) + O(h^{2+\frac{\beta(2-\alpha_1-\alpha_2)}{2}}) = O(h^{2+\frac{\beta(2-\alpha_1-\alpha_2)}{2}}).$$

We see how the situation is complicated only to compute the expectation of $H_{nj}^{'}$. We expect further difficulties when we try to calculate the speed of convergence of the second moment of $H_{nj}^{'}$ to obtain its variance necessary to apply the Lindeberg-Feller theorem.

Chapter 5

Simulation results

"I filosofi hanno solo interpretato il mondo in vari modi,
il punto e' cambiarlo."
(K. Marx)

5.1 First case: the jump component is a Compound Poisson process

We show the performance of the threshold estimator $\tilde{v}_{1,1}^{(n)}(X^{(1)}, X^{(2)})_T$ of the covariation between the two diffusion parts by a simulated model. In particular, we simulate jump diffusion processes with jump components given by compound Poisson processes with Gaussian size of jumps and constant diffusion coefficients

$$dX_t^{(1)} = \sigma^{(1)} dW_t^{(1)} + \sum_{k=1}^{N_T^{(1)}} Z_k^{(1)}$$

and

$$dX_t^{(2)} = \sigma^{(2)} dW_t^{(2)} + \sum_{k=1}^{N_T^{(2)}} Z_k^{(2)}$$

where $Z_k^{(1)} \sim i.i.d.N(0, 0.36)$ and $Z_k^{(2)} \sim i.i.d.N(0, 0.25)$. The parameters of the Poisson processes $N^{(1)}$ and $N^{(2)}$ are $\lambda^{(1)} = 5$ and $\lambda^{(2)} = 6$ respectively, whereas $\sigma^{(1)} = 0.3$, $\sigma^{(2)} = 0.4$ and $\rho = -0.7$.

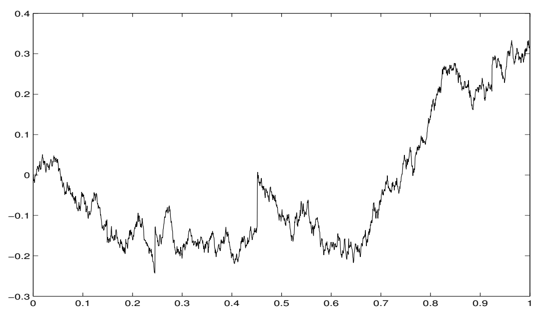

Figure 5.1: Example of sample path of a jump diffusion process with Compound Poisson jumps

To generate 5000 trajectories of each $X^{(q)}$, we fix $T = 1$ and take n equally spaced observations $X_{\frac{j}{n}}$, $j = 1, 2, ..., n$. To evaluate the performance of our estimator we simulate several situations in which n assumes increasing values. If $T = 1$ represents a year if we choose $n = 250, 500, 1000, ...,$

Number of returns (n)	Mean	Median	Standard Deviation	Kurtosis	Skewness
250	0.4218	0.3598	1.0610	3.3346	0.3779
500	0.3110	0.2738	1.0260	3.0216	0.1847
1000	0.2399	0.2081	1.0283	3.2531	0.1640
2000	0.1816	0.1543	1.0184	3.1137	0.2199

Table 5.1: Descriptive statistics relative to N=5000 replications of the jump diffusion model with Compound Poisson jumps.

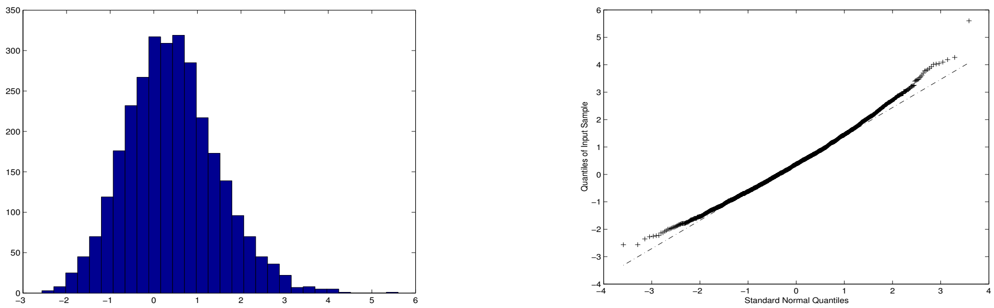

Figure 5.2: Histogram and Normal Probability Plot of 5000 values of $\frac{h^{-1/2}(\tilde{v}^{(n)}_{1,1}(X^{(1)},X^{(2)})_T - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\tilde{v}^{(n)}_{2,2}(X^{(1)},X^{(2)})_T - \tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}}$, with n =250.

we reproduce the situation where we have daily observations, two observations per day and so on. Finally, we choose $r(h) = h^{0.9}$.

In particular, we are interested in values assumed by the quantity $\frac{h^{-1/2}(\tilde{v}^{(n)}_{1,1}(X^{(1)},X^{(2)})_T - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\tilde{v}^{(n)}_{2,2}(X^{(1)},X^{(2)})_T - \tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}}$, to evaluate the performance of the threshold estimator in small sample. Table 5.1 contains the descriptive statistics relative to 5000 replications of the sample path of our model. We observe a moderate bias which tends to decrease as the number of returns n increases.

5.2 Second case: the jump component is a Variance Gamma process

The Variance Gamma process, which we use in this second simulated model, is a purely jumping process with infinite activity and finite variation whose Blumenthal-Gatoor index is 0. The following figure depicts an example of a sample path.

The VG process of parameters κ, θ and ς is obtained by evaluating Brownian motion with drift

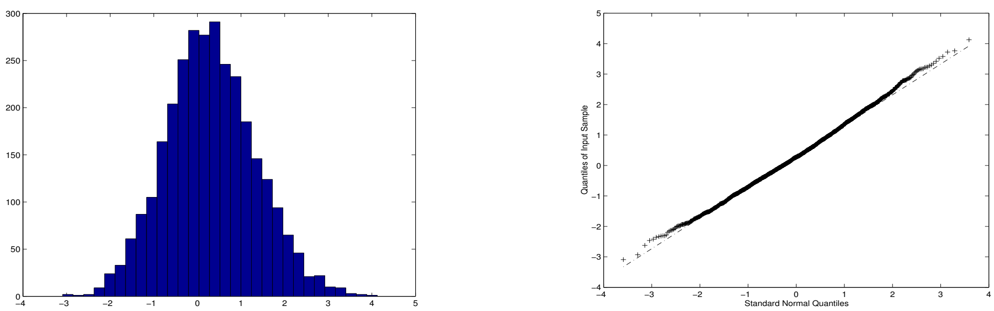

Figure 5.3: Histogram and Normal Probability Plot of 5000 values of $\frac{h^{-1/2}(\tilde{v}^{(n)}_{1,1}(X^{(1)},X^{(2)})_T - \int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\tilde{v}^{(n)}_{2,2}(X^{(1)},X^{(2)})_T - \tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}}$, with n =500.

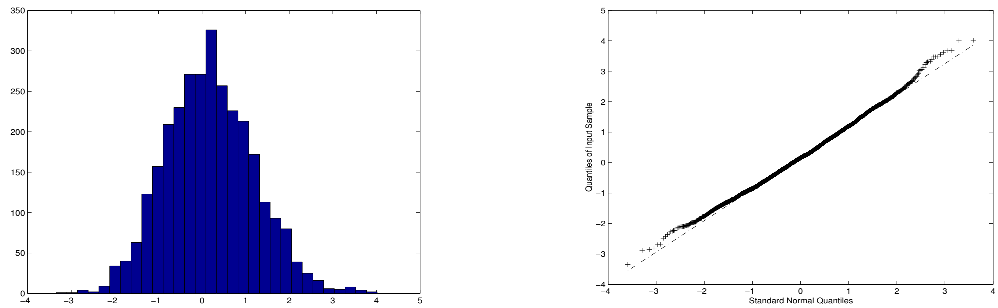

Figure 5.4: Histogram and Normal Probability Plot of 5000 values of $\frac{h^{-1/2}(\tilde{v}_{1,1}^{(n)}(X^{(1)},X^{(2)})_T-\int_0^T \rho\sigma_t^{(1)}\sigma_t^{(2)}dt)}{\sqrt{\tilde{v}_{2,2}^{(n)}(X^{(1)},X^{(2)})_T-\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}}$, with n =2000.

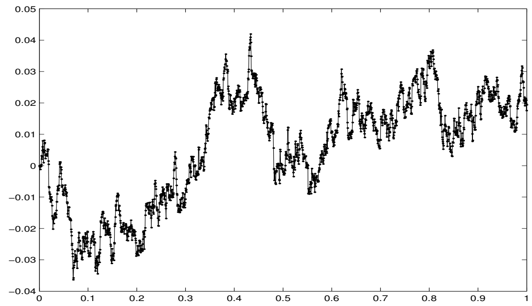

Figure 5.5: Example of sample path of a Variance Gamma process

at a random time given by a gamma process, Γ, which is a process of independent increments distributed as a Gamma r.v. with mean and variance depending on the time interval time h. Formally $VG_t(\kappa,\theta,\varsigma)=W_{\Gamma_t(1,\theta)}(\varsigma,\kappa)$, (Madan, Carr, Chang, 1998). In this simulated model we choose to test the performance of the threshold estimator in two cases: in the first one the jump component is given by a VG process only, in the second one we add the rare jumps of a Compound Poisson process too. The parameters of the $VG^{(1)}$ process relative to the first simulated model $X^{(1)}$ are κ_1 =0.08, θ_1 =0.1, ς_1 =0.03, while for the second $VG^{(2)}$ are κ_2 =0.02, θ_2 =0.08, ς_2 =0.06. Tables 5.2 and 5.3 show the descriptive statistics of the replications in the two cases.

As in preceding section we show a graphical analysis of the simulated results.

5.3 Final remarks

[h] The simulated results presented in this paragraph emphasize how the threshold estimator is characterized by a moderate bias in small samples. In particular, a positive skewness is evident when the number of sampled points n is not large, especially when the jump component contains a Compound Poisson process. However, such property rapidly tends to disappear as n increases.

Number of points	Mean	Median	Standard Deviation	Kurtosis	Skewness
250	0.1067	0.0331	1.0489	3.8937	0.4654
500	0.0928	0.0403	1.0367	3.2456	0.2796
1000	0.0575	0.0351	1.0045	3.1936	0.1711
2000	0.0199	-0.0095	1.0142	2.9879	0.1944

Table 5.2: Descriptive statistics relative to N=5000 replications when the jump part is given by a VG process.

Number of points	Mean	Median	Standard Deviation	Kurtosis	Skewness
250	0.3824	0.3152	1.0515	3.6118	0.4212
500	0.2842	0.2285	1.0191	3.0262	0.2075
1000	0.2071	0.1556	1.0497	2.9948	0.2148
2000	0.1624	0.1410	1.0104	2.9946	0.0473

Table 5.3: Descriptive statistics relative to N=5000 replications when the in the jump part are both VG and Compound Poisson processes.

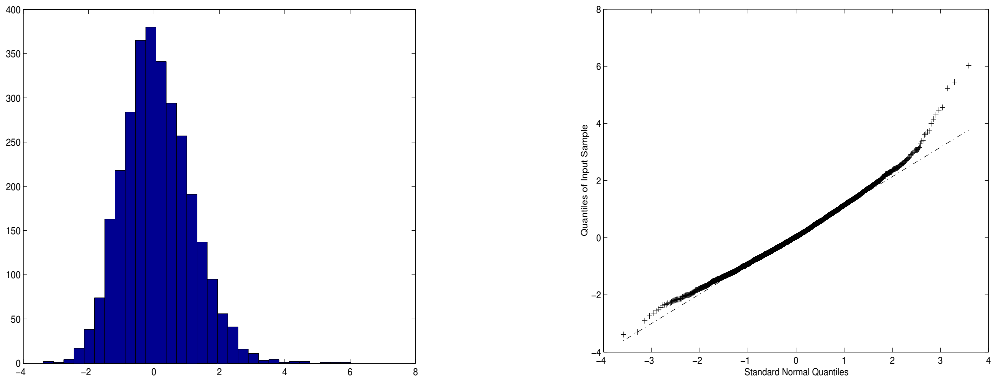

Figure 5.6: Histogram and Normal Probability Plot of 5000 values of $\frac{h^{-1/2}(\tilde{v}^{(n)}_{1,1}(X^{(1)},X^{(2)})_T-\int_0^T \rho\sigma^{(1)}_t\sigma^{(2)}_t dt)}{\sqrt{\tilde{v}^{(n)}_{2,2}(X^{(1)},X^{(2)})_T-\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}}$, with n =250 and jump part given by a VG process only.

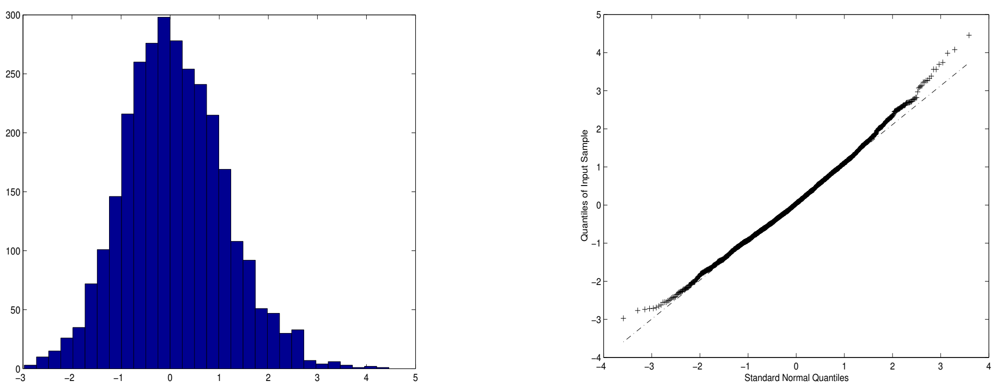

Figure 5.7: Histogram and Normal Probability Plot of 5000 values of $\frac{h^{-1/2}(\tilde{v}^{(n)}_{1,1}(X^{(1)},X^{(2)})_T-\int_0^T \rho\sigma^{(1)}_t\sigma^{(2)}_t dt)}{\sqrt{\tilde{v}^{(n)}_{2,2}(X^{(1)},X^{(2)})_T-\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}}$, with n =500 and jump part given by a VG process only.

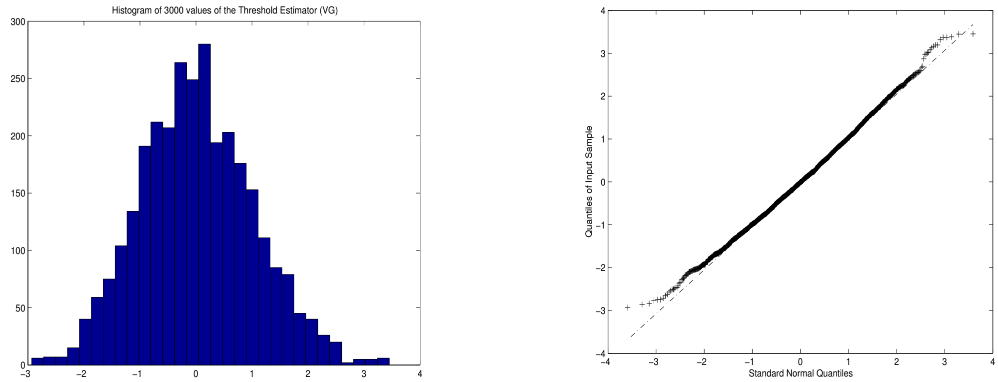

Figure 5.8: Histogram and Normal Probability Plot of 5000 values of $\frac{h^{-1/2}(\tilde{v}^{(n)}_{1,1}(X^{(1)},X^{(2)})_T-\int_0^T \rho\sigma^{(1)}_t\sigma^{(2)}_t dt)}{\sqrt{\tilde{v}^{(n)}_{2,2}(X^{(1)},X^{(2)})_T-\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}}$, with n =2000 and jump part given by a VG process only.

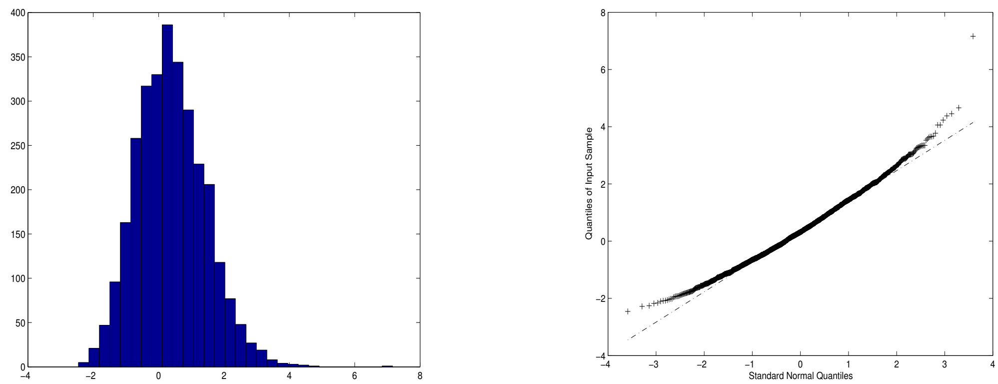

Figure 5.9: Histogram and Normal Probability Plot of 5000 values of $\frac{h^{-1/2}(\tilde{v}^{(n)}_{1,1}(X^{(1)},X^{(2)})_T-\int_0^T \rho\sigma^{(1)}_t\sigma^{(2)}_t dt)}{\sqrt{\tilde{v}^{(n)}_{2,2}(X^{(1)},X^{(2)})_T-\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}}$, with n =250 and jump part given by a VG process plus a Compound Poisson.

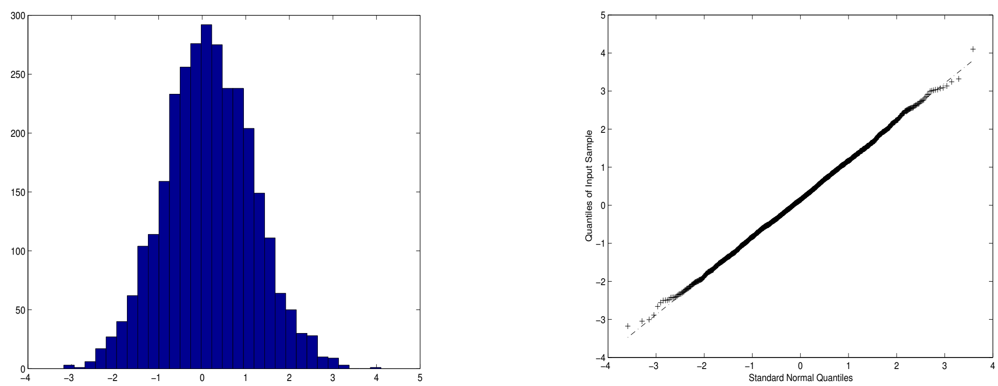

Figure 5.10: Histogram and Normal Probability Plot of 5000 values of $\frac{h^{-1/2}(\tilde{v}^{(n)}_{1,1}(X^{(1)},X^{(2)})_T-\int_0^T \rho\sigma^{(1)}_t\sigma^{(2)}_t dt)}{\sqrt{\tilde{v}^{(n)}_{2,2}(X^{(1)},X^{(2)})_T-\tilde{w}^{(n)}(X^{(1)},X^{(2)})_T}}$, with n =2000 and jump part given by a VG process plus a Compound Poisson.

Moreover, in every case the mean of the simulated values of $\tilde{v}_{1,1}^{(n)}(X^{(1)}, X^{(2)})_T$ is very close to the theoretical integrated covariation $\rho\sigma^{(1)}\sigma^{(2)}$. The median of the normalized version of the threshold estimator is always less than the mean whereas the skewness approaches to zero (as n increases) more quickly in the case where the jump component is given by a Variance Gamma process only.

Bibliography

ANDERSEN, T.G., BOLLERSLEV, T. (1998): *Answering the skeptics: yes standard volatility models to provide accuracy forecasts. International Economic Review*, *39*, 885-905.

item[] ANDERSEN, T.G., BOLLERSLEV, T., DIEBOLD, F.X. (2005): *Roughing it up: Including jump components in the measurement, modeling and forecasting of return volatility.* Manuscript, Northwestern University, Duke University and University of Pennsylvania.

ANDERSEN, T.G., BOLLERSLEV, T., DIEBOLD, F.X., LABYS, P. (2003): *Modelling and forecasting realized volatility. Econometrica*, *71*, 579-625.

BALDI, P. (2000): *Equazioni Differenziali Stocastiche e Applicazioni.* Pitagora Editrice. Bologna 2000.

BARNDORFF-NIELSEN, O.E., SHEPHARD, N. (2003): *Realised power variation and stochastic volatility. Bernouilli*, *9*, 243-265.

BARNDORFF-NIELSEN, O.E., SHEPHARD, N. (2004a): *Power and bipower variation with stochastic volatility and jumps, (with discussion). Journal of Financial Econometrics*, *2*, 1-48.

BARNDORFF-NIELSEN, O.E., SHEPHARD, N. (2004b): *A feasible central limit theory for realized volatility under leverage.* working paper.

BARNDORFF-NIELSEN, O.E., SHEPHARD, N. (2004c): *Econometric analysis of realized covariation: high frequency based covariance, regression and correlation in financial economics. Econometrica*, (2004), *72*, 885-925.

BARNDORFF-NIELSEN, O.E., GRAVENSEN, S.E., JACOD, J., PODOLSKIJ, M., SHEPHARD, N. (2005): *A Central limit theorem for realized power and bipower variations of continuous semimartingales.* working paper.

BERTOIN, J. (1996): *Lévy processes.* Cambridge University Press: Cambridge.

BJORK, T. (1998): *Arbitrage Theory in Continuous Time.* Oxford University Press.

BOLLERSLEV, T. (2001): *Financial Econometrics: Past developments and future challenges. Journal of Econometrics*, 100, 41-51.

CARR, P., GEMAN, H., MADAN, D., YOR, M. (2002): *The finite structure of asset returns: An empirical investigation. Journal of Business*, 75.

CARR, P., GEMAN, H., MADAN, D., YOR, M. (2003): *Stochastic Volatility for Lévy Processes. Mathematical Finance*, 13, 345-382.

CARTER, M., VAN BRUNT, B. (2000): *The Lebesgue-Stilties integral. A pratical introduction.* Springer.

CHERUBINI, U., LUCIANO, E., VECCHIATO, W. (2004): *Copula Methods in Finance.* John Wiley and Sons, Ltd.

CHUNG, K. L., WILLIAMS, R. J. (1990): *Introduction to Stochastic Integration*. Probability and Its Applications. Birkhauser.

COMTE, O.E., RENAULT, N. (1998): *Long memory in continuous-time stochastic volatility models. Mathematical Finance*, *8*, 291-323.

CONT, R. (2001): *Empirical properties of asset returns: Stylized facts and statistical issue. Quantitative Finance*, *1*, 1-14.

CONT, R. (2005): *Model uncertainty and its impact on the pricing of derivative instruments.* To appear in *Mathematical Finance*.

CONT, R., MANCINI, C. (2005): *Detecting the presence of a diffusion and the nature of the jumps in asset prices.* working paper.

CONT, R., TANKOV, P. (2004): *Financial Modelling with jump processes.* Chapman and Hall-CRC.

ETHIER, S.N., KURTZ, T.G. (1986): *Markov processes: characterization and convergence.* Wiley, NY.

GEMAN, H., MADAN, D., YOR, M. (2001): *Time changes for Lévy Processes. Mathematical Finance*, 11, 79-96.

GIHMAM, I.I., SKOROHOD, A.V. (1965): *Introduction to the Theory of Random Processes.* W. B. Saunders Company.

HAYASHI, T., YOSHIDA, N. (2006): *On covariance estimationof Non-synchronously observed diffusion processes. Bernouilli*, 11-2, 359-379.

HARRISON, J.M., KREPS, D. (1979): *Martingales and arbitrage in multiperiod security markets. Journal of Economic Theory*, 20, 381-408.

IKED, N., WATANABE, S. (1981): *Stochastic Differential Equations and Diffusion processes.* North Holland.

JACOD, J., SHIRYAEV, A.N. (1987): *Limit Theorems for Stochastic Processes.* Sringer-Verlag, Berlin.

JOHANNES, M. (2004): *The statistical and economic role of jumps in continuous time interest models . The Journal of Finance*, 59, 227-260.

KALLENBERG, O. (1997): *Foundation of Modern Probability.* Springer: New York.

KLEBANER, F.C. (1999): *Introduction to Stochastic Calculus with Applications.* Imperial College Press.

KRYLOV, N.V. (2002): *Introduction to the Theory of Random Processes.* Graduate Studies in Mathematics, Volume 43, American Mathematical Society.

LOEVE, M. (1960): *Probability Theory.* Second Edition. D. Van Nostrand Company, Inc.

MANCINI, C. (2004): *Estimation of the parameters of jump of a general Poisson diffusion model. Scandinavian Actuarial Journal*, *1*, 42–52.

MANCINI, C. (2005): *Estimating the integrated volatility in stochastic volatility models with Lévy type jumps.* Pre-print, Dipartmento di Matematica per le Decisioni, Universita' di Firenze.

MANCINI, C., RENO', R. (2005): *Threshold estimator of jump-diffusion models in interest rate modelling.* working paper

MERTON, R. (1976): *Option pricing when underlying stock returns are discontinuous. Journal of Financial Economics*, *3*, 125-144.

METIVIER, M. (1982): *Semimartingales: a course on stochastic processes.* De Gruyter.

PAGAN, A. (1996): *The Econometrics of Financial Markets. Journal of Empirical Finance*, 3, 15-102.

PROTTER, P. (1990): *Stochastic integration and Differential equation, a new approach.* Springer-Verlag.

REVUZ, D., YOR, M. (1999): *Continuous martingales and Brownian motion.* Springer-Verlag.

SATO, K. (1999): *Lévy processes and Infinitely Divisible Distributions.* Cambridge Studies in Advanced Mathematics.

SHIRYAYEV, A.N. (1981): *Martingales: recent developments, results and applications. International Statistical Review*, *49*, 199-233.

WOERNER, J. (2003): *Estimating of integrated volatility in stochastic volatility models.* Unpublished paper: OCIAM, Mathematical Institute, University of Oxford.